HL Paper 3

Consider the recurrence relation $au_{n+2} + bu_{n+1} + cu_n = 0$, $n \in \mathbb{N}$ where a, b and c are constants. Let α and β denote the roots of the equation $ax^2 + bx + c = 0$.

a. Verify that the recurrence relation is satisfied by [4]

$$u_n = A\alpha^n + B\beta^n,$$

where A and B are arbitrary constants.

b. Solve the recurrence relation [9]

$u_{n+2} - 2u_{n+1} + 5u_n = 0$ given that $u_0 = 0$ and $u_1 = 4$.

Markscheme

a. attempt to substitute the given expression for u_n into the recurrence relation   ***M1***

$au_{n+2} + bu_{n+1} + cu_n = a(A\alpha^{n+2} + B\beta^{n+2}) + b(A\alpha^{n+1} + B\beta^{n+1}) + c(A\alpha^n + B\beta^n)$ ***A1***

$= A\alpha^n(a\alpha^2 + b\alpha + c) + B\beta^n(a\beta^2 + b\beta + c)$ ***A1***

$= 0$ because α and β both satisfy $ax^2 + bx + c = 0$ ***R1AG***

Note: Award ***M1A0A1R0*** for solutions that are set to zero throughout and conclude with $0 = 0$. Award the ***R1*** for any valid reason.

[4 marks]

b. the auxiliary equation is $x^2 - 2x + 5 = 0$ ***A1***

solving their quadratic equation   ***(M1)***

the roots are $1 \pm 2\mathrm{i}$ ***A1***

the general solution is

$u_n = A(1 + 2\mathrm{i})^n + B(1 - 2\mathrm{i})^n$ $\left(u_n = \left(\sqrt{5}\right)^n (A\operatorname{cis}(n\arctan 2) + B\operatorname{cis}(-n\arctan 2)) \right)$ ***(A1)***

attempt to substitute both boundary conditions   ***M1***

$A + B = 0$; $A(1 + 2\mathrm{i}) + B(1 - 2\mathrm{i}) = 4$ ***A1***

attempt to solve their equations for A and B ***M1***

$A = -\mathrm{i}$, $B = \mathrm{i}$ ***A1***

$u_n = \mathrm{i}(1 - 2\mathrm{i})^n - \mathrm{i}(1 + 2\mathrm{i})^n$ $\left(u_n = 2\left(\sqrt{5}\right)^n \sin(n\arctan 2) \right)$ ***A1***

Note: Accept the trigonometric form for u_n.

[9 marks]

Examiners report

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a. Show that $\gcd\left(4k+2,\ 3k+1\right) = \gcd\left(k-1,\ 2\right)$, where $k \in \mathbb{Z}^{+}$, $k > 1$. [4]

b.i. State the value of $\gcd\left(4k+2,\ 3k+1\right)$ for odd positive integers k. [1]

b.ii. State the value of $\gcd\left(4k+2,\ 3k+1\right)$ for even positive integers k. [1]

Markscheme

a. **METHOD 1**

attempting to use the Euclidean algorithm **M1**

$4k+2 = 1\left(3k+1\right) + \left(k+1\right)$ **A1**

$3k+1 = 2\left(k+1\right) + \left(k-1\right)$ **A1**

$K+1 = \left(k-1\right) + 2$ **A1**

$= \gcd\left(k-1,\ 2\right)$ **AG**

METHOD 2

$\gcd\left(4k+2,\ 3k+1\right)$

$= \gcd\left(4k+2 - \left(3k+1\right),\ 3k+1\right)$ **M1**

$= \gcd\left(3k+1,\ k+1\right)\ \left(= \gcd\left(k+1, 3k+1\right)\right)$ **A1**

$= \gcd\left(3k+1 - 2\left(k+1\right),\ k+1\right)\ \left(= \gcd\left(k-1, k+1\right)\right)$ **A1**

$= \gcd\left(k+1 - \left(k-1\right),\ k-1\right)\ \left(= \gcd\left(2, k-1\right)\right)$ **A1**

$= \gcd\left(k-1,\ 2\right)$ **AG**

[4 marks]

b.i. (for k odd), $\gcd\left(4k+2,\ 3k+1\right) = 2$ **A1**

[1 mark]

b.ii. (for k even), $\gcd\left(4k+2,\ 3k+1\right) = 1$ **A1**

[1 mark]

Examiners report

The weights of the edges in the complete graph G are given in the following table.

	A	B	C	D	E	F
A	–	4	9	8	14	6
B	4	–	1	14	9	3
C	9	1	–	5	12	2
D	8	14	5	–	11	12
E	14	9	12	11	–	7
F	6	3	2	12	7	–

a. Starting at A , use the nearest neighbour algorithm to find an upper bound for the travelling salesman problem for G. [5]

b. By first deleting vertex A , use the deleted vertex algorithm together with Kruskal's algorithm to find a lower bound for the travelling salesman [7]

problem for G.

Markscheme

a. the edges are traversed in the following order

AB   *A1*

BC

CF   *A1*

FE

ED   *A1*

DA *A1*

upper bound $=$ weight of this cycle $= 4 + 1 + 2 + 7 + 11 + 8 = 33$ *A1*

[5 marks]

b. having deleted A, the order in which the edges are added is

BC   *A1*

CF   *A1*

CD   *A1*

EF   *A1*

Note: Accept indication of the correct order on a diagram.

to find the lower bound, we now reconnect A using the two edges with the lowest weights, that is AB and AF   *(M1)(A1)*

lower bound $= 1 + 2 + 5 + 7 + 4 + 6 = 25$ *A1*

Note: Award *(M1)(A1)A1* for $\text{LB} = 15 + 4 + 6 = 25$ obtained either from an incorrect order of correct edges or where order is not indicated.

Examiners report

a. [N/A]

b. [N/A]
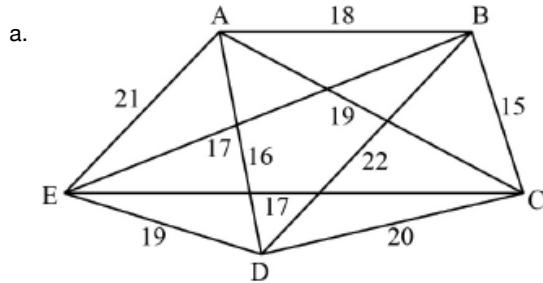
Mathilde delivers books to five libraries, A, B, C, D and E. She starts her deliveries at library D and travels to each of the other libraries once, before returning to library D. Mathilde wishes to keep her travelling distance to a minimum.

The weighted graph H, representing the distances, measured in kilometres, between the five libraries, has the following table.

	A	B	C	D	E
A	–	18	19	16	21
B	18	–	15	22	17
C	19	15	–	20	17
D	16	22	20	–	19
E	21	17	17	19	–

a. Draw the weighted graph H. [2]

b. Starting at library D use the nearest-neighbour algorithm, to find an upper bound for Mathilde's minimum travelling distance. Indicate clearly the order in which the edges are selected. [5]

c. By first removing library C, use the deleted vertex algorithm, to find a lower bound for Mathilde's minimum travelling distance. [4]

Markscheme

a.

complete graph on 5 vertices   *A1*

weights correctly marked on graph   *A1*

[2 marks]

b. clear indication that the nearest-neighbour algorithm has been applied   *M1*

DA (or 16)   *A1*

AB (or 18) then BC (or 15)   *A1*

CE (or 17) then ED (or 19)   *A1*

$$\text{UB} = 85 \quad \textbf{\textit{A1}}$$

[5 marks]

c. an attempt to find the minimum spanning tree   *(M1)*

DA (16) then BE (17) then AB (18) (total 51)   *A1*

reconnect C with the two edges of least weight, namely CB (15) and CE (17)   *M1*

$\text{LB} = 83 \quad$ *A1*

[4 marks]

Examiners report

A recurrence relation is given by $u_{n+1} + 2u_n + 1 = 0, \ u_1 = 4$.

a. Use the recurrence relation to find u_2.   [1]

b. Find an expression for u_n in terms of n.   [6]

c. A second recurrence relation, where $v_1 = u_1$ and $v_2 = u_2$, is given by $v_{n+1} + 2v_n + v_{n-1} = 0, \ n \geq 2$.   [6]

Find an expression for v_n in terms of n.

Markscheme

a. $u_2 = -9 \quad$ *A1*

[1 mark]

b. **METHOD 1**

$u_{n+1} = -2u_n - 1$

let $u_n = a(-2)^n + b \quad$ *M1A1*

EITHER

$a(-2)^{n+1} + b = -2\left(a(-2)^n + b\right) - 1 \quad$ *M1*

$a(-2)^{n+1} + b = a(-2)^{n+1} - 2b - 1$

$3b = -1$

$b = -\frac{1}{3} \quad$ *A1*

$u_1 = 4 \Rightarrow -2a - \frac{1}{3} = 4 \quad$ *(M1)*

$\Rightarrow a = -\frac{13}{6} \quad$ *A1*

OR

using $u_1 = 4, \ u_2 = -9$

$4 = -2a + b, \ -9 = 4a + b \quad$ *M1A1*

solving simultaneously **M1**

$\Rightarrow a = -\frac{13}{6}$, and $b = -\frac{1}{3}$ **A1**

THEN

so $u_n = -\frac{13}{6}(-2)^n - \frac{1}{3}$

METHOD 2

use of the formula $u_n = a^n u_0 + b\left(\frac{1-a^n}{1-a}\right)$ **(M1)**

letting $u_0 = -\frac{5}{2}$ **A1**

letting $a = -2$ and $b = -1$ **A1**

$u_n = -\frac{5}{2}(-2)^n - 1\left(\frac{1-(-2)^n}{1--2}\right)$ **M1A1**

$= -\frac{13}{6}(-2)^n - \frac{1}{3}$ **A1**

[6 marks]

c. auxiliary equation is $k^2 + 2k + 1 = 0$ **M1**

hence $k = -1$ **(A1)**

so let $v_n = (an + b)(-1)^n$ **M1**

$(2a + b)(-1)^2 = -9$ and $(a + b)(-1)^1 = 4$ **A1**

so $a = -5$, $b = 1$ **M1A1**

$v_n = (1 - 5n)(-1)^n$

Note: Caution necessary to allow FT from (a) to part (c).

[6 marks]

Total [13 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Consider κ_n, a complete graph with n vertices, $n \geqslant 2$. Let T be a fixed spanning tree of κ_n.

a.i. Draw the complete bipartite graph $\kappa_{3,3}$. [1]

a.ii.Prove that $\kappa_{3,3}$ is not planar. [4]

b. A connected graph G has v vertices. Prove, using Euler's relation, that a spanning tree for G has $v - 1$ edges. [2]

c. If an edge E is chosen at random from the edges of κ_n, show that the probability that E belongs to T is equal to $\frac{2}{n}$. [4]

Markscheme

a.i.

A1

[1 mark]

a.ii. assume $\kappa_{3,3}$ is planar

$\kappa_{3,3}$ has no cycles of length 3   **R1**

use of $e \leqslant 2v - 4$ **M1**

$e = 9$ and $v = 6$ **A1**

hence inequality not satisfied $9 \not\leqslant 8$ **R1**

so $\kappa_{3,3}$ is not planar   **AG**

Note: use of $e \leqslant 3v - 6$ with $e = 9$ and $v = 6$ and concluding that this inequality does not show whether $\kappa_{3,3}$ is planar or not just gains **R1**.

[4 marks]

b. a spanning tree (is planar and) has one face   **A1**

Euler's relation is $v - e + f = 2$

$v - e + 1 = 2$ **M1**

$\Rightarrow e = v - 1$ **AG**

[2 marks]

c. κ_n has $\begin{pmatrix} n \\ 2 \end{pmatrix}$ edges $\left(\frac{n(n-1)}{2} \text{ edges} \right)$ **(A1)**

$\mathrm{P}(E \text{ belongs to } T) = \dfrac{n-1}{\left(\frac{n(n-1)}{2} \right)}$ **M1A1**

clear evidence of simplification of the above expression   **M1**

$= \frac{2}{n}$ **AG**

[4 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]
c. [N/A]

The simple, complete graph $\kappa_n (n > 2)$ has vertices $\mathrm{A}_1,\ \mathrm{A}_2,\ \mathrm{A}_3,\ \ldots,\ \mathrm{A}_n$. The weight of the edge from A_i to A_j is given by the number $i + j$.

Consider the general graph κ_n.

a. (i)   Draw the graph κ_4 including the weights of all the edges.   [4]
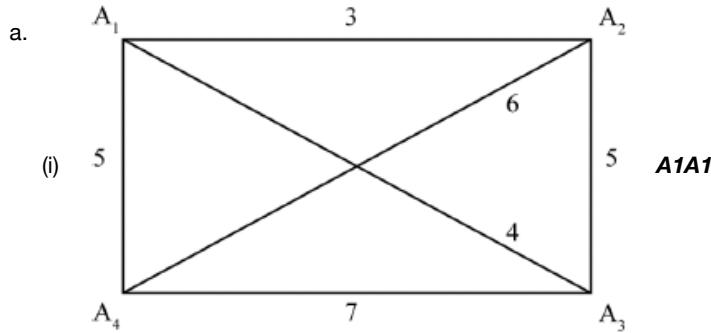
(ii)   Use the nearest-neighbour algorithm, starting at vertex A_1, to find a Hamiltonian cycle.

(iii)   Hence, find an upper bound to the travelling salesman problem for this weighted graph.

b. Consider the graph κ_5. Use the deleted vertex algorithm, with A_5 as the deleted vertex, to find a lower bound to the travelling salesman   [5]

   problem for this weighted graph.

c. (i)   Use the nearest-neighbour algorithm, starting at vertex A_1, to find a Hamiltonian cycle.   [7]

   (ii)   Hence find and simplify an expression in n, for an upper bound to the travelling salesman problem for this weighted graph.

d. By splitting the weight of the edge A_iA_j into two parts or otherwise, show that all Hamiltonian cycles of κ_n have the same total weight, equal to [3]

   the answer found in (c)(ii).

Markscheme

a.

(i)

A1A1

Note: A1 for the graph, **A1** for the weights.

   (ii)   cycle is $A_1A_2A_3A_4A_1$ **A1**

   (iii)   upper bound is $3 + 5 + 7 + 5 = 20$ **A1**

   [4 marks]

b. with A_5 deleted, (applying Kruskal's Algorithm) the minimum spanning tree will consist of the edges A_1A_2, A_1A_3, A_1A_4, of weights 3, 4, 5

   (M1)A1

   the two edges of smallest weight from A_5 are A_5A_1 and A_5A_2 of weights 6 and 7   **(M1)A1**

   so lower bound is $3 + 4 + 5 + 6 + 7 = 25$ **A1**

   [5 marks]

c. (i)   starting at A_1 we go A_2, A_3 ... A_n

   we now have to take A_nA_1

   thus the cycle is $A_1A_2A_3 \ldots A_{n-1}A_nA_1$ **A1A1**

   Note: Final **A1** is for A_nA_1.

   (ii)   smallest edge from A_1 is A_1A_2 of weight 3, smallest edge from A_2 (to a new vertex) is A_2A_3 of weight 5, smallest edge from A_{n-1} (to a new vertex) is $A_{n-1}A_n$ of weight $2n - 1$ **(M1)**

   weight of A_nA_1 is $n + 1$

   weight is $3 + 5 + 7 + \ldots + (2n - 1) + (n + 1)$ **A1**

   $= \frac{(n-1)}{2}(2n + 2) + (n + 1)$ **M1A1**

   $= n(n + 1)$ (which is an upper bound)   **A1**

[7 marks]

d. put a marker on each edge $A_i A_j$ so that i of the weight belongs to vertex A_i and j of the weight belongs to vertex A_j *M1*

the Hamiltonian cycle visits each vertex once and only once and for vertex A_i there will be weight i (belonging to vertex A_i) both going in and coming out   *R1*

so the total weight will be $\sum\limits_{i=1}^{n} 2i = 2\frac{n}{2}(n+1) = n(n+1)$ *A1AG*

Note: Accept other methods for example induction.

[3 marks]

Examiners report

In this question the notation $(a_n a_{n-1} \ldots a_2 a_1 a_0)_b$ is used to represent a number in base b, that has unit digit of a_0. For example $(2234)_5$ represents

$2 \times 5^3 + 2 \times 5^2 + 3 \times 5 + 4 = 319$ and it has a unit digit of 4.

Let x be the cube root of the base 7 number $(503231)_7$.

a. (i)   By converting the base 7 number to base 10, find the value of x, in base 10.   [7]

(ii)   Express x as a base 5 number.

b. Let y be the base 9 number $(a_n a_{n-1} \ldots a_1 a_0)_9$. Show that y is exactly divisible by 8 if and only if the sum of its digits, $\sum\limits_{i=0}^{n} a_i$, is also exactly   [7]
divisible by 8.

c. Using the method from part (b), find the unit digit when the base 9 number $(321321321)_9$ is written as a base 8 number.   [3]

Markscheme

a. (i)   converting to base 10

$(503231)_7 = 5 \times 7^5 + 3 \times 7^3 + 2 \times 7^2 + 3 \times 7 + 1 = 85184$ *M1A1A1*

so $x = 44$ *A1*

(ii)   repeated division by 5 gives   *(M1)*

44	remainder
8	4
1	3
	1

so base 5 value for x is $(134)_5$ *A1*

Notes: Alternative method is to successively subtract the largest multiple of 25 and then 5.

Follow through if they forget to take the cube root and obtain $(10211214)_5$ then award *(M1)(A1)A1*.

[7 marks]

b. $9 \equiv 1(\bmod 8)$ *A1*

$9^i \equiv 1^i \equiv 1(\bmod 8) \quad i \in \mathbb{N}$ *(M1)(A1)*

$y = a_n 9^n + a_{n-1} 9^{n-1} + \ldots + a_1 9 + a_0 \equiv a_n 1^n + a_{n-1} 1^{n-1} + \ldots + a_1 1 + a_0 \equiv$

$a_n + a_{n-1} + \ldots + a_1 + a_0 \equiv \sum_{i=0}^{n} a_i (\bmod 8)$ *M1A1A1*

so $y = 0(\bmod 8)$ and hence divisible by 8 if and only if $\sum_{i=0}^{n} a_i \equiv 0(\bmod 8)$ and hence divisible by 8   *R1AG*

Note: Accept alternative valid methods *eg* binomial expansion of $(8+1)^i$, factorization of $(a^i - 1)$ if they have sufficient explanation.

[7 marks]

c. using part (b), $(321321321)_9 \equiv 3 + 2 + 1 + 3 + 2 + 1 + 3 + 2 + 1 = 18 \equiv 2(\bmod 8)$ *M1A1*

so the unit digit is 2   *A1*

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Consider the system of linear congruences

$$x \equiv 2(\bmod 5)$$
$$x \equiv 5(\bmod 8)$$
$$x \equiv 1(\bmod 3).$$

a. With reference to the integers 5, 8 and 3, state why the Chinese remainder theorem guarantees a unique solution modulo 120 to this system of   [2]

linear congruences.

b. Hence or otherwise, find the general solution to the above system of linear congruences.   [7]

Markscheme

a. 5, 8 and 3 are pairwise relatively prime (or equivalent)   *R1*

120 is the product of 5, 8 and 3   *R1*

b. **METHOD 1**

$x = 2 + 5t,\ t \in \mathbb{Z}$ and $2 + 5t \equiv 5(\ \mathrm{mod}\ 8)$ ___M1___

$5t \equiv 3(\ \mathrm{mod}\ 8) \Rightarrow 5t \equiv 35(\ \mathrm{mod}\ 8)$ ___(M1)___

$t = 7 + 8u,\ u \in \mathbb{Z}$ ___(A1)___

$x = 2 + 5(7 + 8u) \Rightarrow x = 37 + 40u$ ___(A1)___

$37 + 40u \equiv 1(\ \mathrm{mod}\ 3) \Rightarrow u \equiv 0(\ \mathrm{mod}\ 3)$ ___(A1)___

$u = 3v,\ v \in \mathbb{Z}$ ___(A1)___

$x = 37 + 40(3v)$

$x = 37 + 120v\ (x \equiv 37(\ \mathrm{mod}\ 120))$ ___A1___

METHOD 2

attempting systematic listing of possibilities ___M1___

solutions to $x \equiv 2(\ \mathrm{mod}\ 5)$ are $x = 2,\ 7,\ 12,\ \ldots,\ 37,\ \ldots$ ___(A1)___

solutions to $x \equiv 5(\ \mathrm{mod}\ 8)$ are $x = 5,\ 13,\ 21,\ \ldots,\ 37,\ \ldots$ ___(A1)___

solutions to $x \equiv 1(\ \mathrm{mod}\ 3)$ are $x = 1,\ 4,\ 7,\ \ldots,\ 37,\ \ldots$ ___(A1)___

a solution is $x = 37$ ___(A1)___

using the Chinese remainder theorem ___(M1)___

$x = 37 + 120v\ (x \equiv 37(\ \mathrm{mod}\ 120))$ ___A1___

METHOD 3

attempting to find $M_i,\ i = 1,\ 2,\ 3$

$M_1 = 8 \times 3 = 24,\ M_2 = 5 \times 3 = 15$ and $M_3 = 5 \times 8 = 40$ ___M1___

using $M_i x_i \equiv 1(\ \mathrm{mod}\ m_i),\ i = 1,\ 2,\ 3$ to obtain

$24x_1 \equiv 1(\ \mathrm{mod}\ 5),\ 15x_2 \equiv 1(\ \mathrm{mod}\ 8)$ and $40x_3 \equiv 1(\ \mathrm{mod}\ 3)$ ___M1___

$x_1 \equiv 4(\ \mathrm{mod}\ 5),\ x_2 \equiv 7(\ \mathrm{mod}\ 8)$ and $x_3 \equiv 1(\ \mathrm{mod}\ 3)$ ___(A1)(A1)(A1)___

use of $x \equiv a_1 x_1 M_1 + a_2 x_2 M_2 + a_3 x_3 M_3(\ \mathrm{mod}\ M)$ gives

$x = (2 \times 4 \times 24 + 5 \times 7 \times 15 + 1 \times 1 \times 40)(\ \mathrm{mod}\ 120)$ ___(M1)___

$x \equiv 757(\ \mathrm{mod}\ 120)\ (\equiv 37(\ \mathrm{mod}\ 120))\ (x = 37 + 120v,\ v \in \mathbb{Z})$ ___A1___

[7 marks]

Examiners report

a. [N/A]

b. [N/A]

In this question no graphs are required to be drawn. Use the handshaking lemma and other results about graphs to explain why,

a. a graph cannot exist with a degree sequence of $1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9$; [2]

b. a simple, connected, planar graph cannot exist with a degree sequence of 4, 4, 4, 4, 5, 5; [3]

c. a tree cannot exist with a degree sequence of 1, 1, 2, 2, 3, 3. [3]

Markscheme

a. assume such a graph exists

by the handshaking lemma the sum of the degrees equals twice the number of edges **R1**

but $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ which is odd **R1**

this is a contradiction so graph does not exist **AG**

[2 marks]

b. assume such a graph exists

since the graph is simple and connected (and $v = 6 > 2$) then $e \leqslant 3v - 6$ **R1**

by the handshaking lemma $4 + 4 + 4 + 4 + 5 + 5 = 2e$ so $e = 13$ **A1**

hence $13 \leqslant 3 \times 6 - 6 = 12$ **A1**

this is a contradiction so graph does not exist **AG**

[3 marks]

c. assume such a graph exists

a tree with 6 vertices must have 5 edges (since $V - E + 1 = 2$ for a tree) **R1**

using the Handshaking Lemma $1 + 1 + 2 + 2 + 3 + 3 = 2 \times 5$ implies $12 = 10$ **M1A1**

this is a contradiction so graph does not exist **AG**

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

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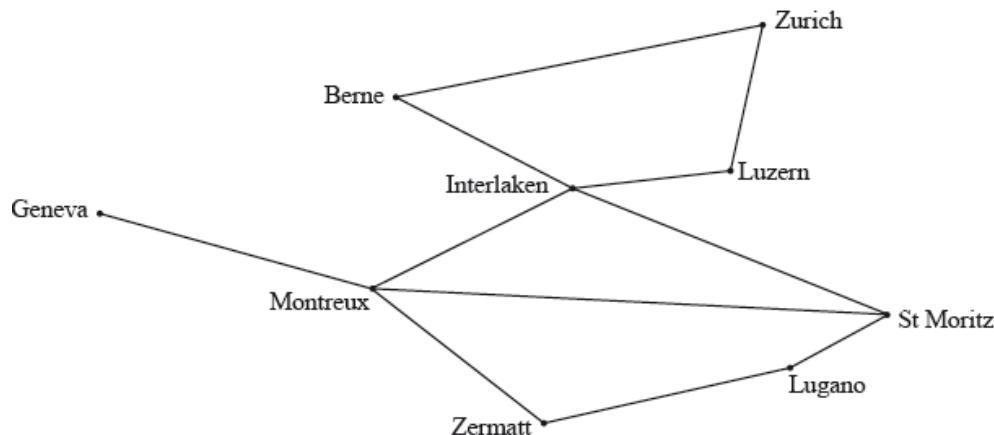
a. The distances by road, in kilometres, between towns in Switzerland are shown in the following table. [5]

	Basel	Berne	Geneva	Lugano	Sion	Zurich
Basel		100	260	260	250	85
Berne	100		170	275	155	125
Geneva	260	170		440	160	290
Lugano	260	275	440		255	210
Sion	250	155	160	255		275
Zurich	85	125	290	210	275	

A cable television company wishes to connect the six towns placing cables along the road system.

Use Kruskal's algorithm to find the minimum length of cable needed to connect the six towns.

b. Visitors to Switzerland can visit some principal locations for tourism by using a network of scenic railways as represented by the following graph: [6]

(i)   State whether the graph has any Hamiltonian paths or Hamiltonian cycles, justifying your answers.

(ii)   State whether the graph has any Eulerian trails or Eulerian circuits, justifying your answers.

(iii)   The tourist board would like to make it possible to arrive in Geneva, travel all the available scenic railways, exactly once, and depart from Zurich. Find which locations would need to be connected by a further scenic railway in order to make this possible.

Markscheme

a. Zurich-Basel   85   *M1A1*

Berne-Basel   100   *A1*

Sion-Berne   155

Sion-Geneva   160

Zurich-Lugano   210   *A1*

total length of pipe needed is 710 km   *A1*

Note: Award *M1* for attempt to start with smallest length.

Note: Accept graphical solution showing lengths chosen.

Note: Award *N1* for a correct spanning tree and *N1* for 710 km with no method shown.

[6 marks]

b. (i)   possible Hamiltonian path *eg* Geneva-Montreux-Zermatt-Lugano-St Moritz-Interlaken-Luzern-Zurich-Berne   *A1*

no possible Hamiltonian cycles *eg* we would have to pass through Montreux twice as Geneva is only connected to Montreux or Interlaken twice   *R1*

(ii)   possible Eulerian trail as there are 2 odd vertices (Geneva and St Moritz)   *R1*

no possible Eulerian circuits as not all even vertices   *R1*

Note: Accept an example of a Eulerian trail for the first *R1*.

(iii)   St Moritz to Zurich   *A2*

[6 marks]

Total [11 marks]

Examiners report

a. [N/A]
b. [N/A]

In a computer game, Fibi, a magic dragon, is climbing a very large staircase. The steps are labelled 0, 1, 2, 3 … .

She starts on step 0. If Fibi is on a particular step then she can either jump up one step or fly up two steps. Let u_n represent the number of different ways that Fibi can get to step n. When counting the number of different ways, the order of Fibi's moves matters, for example jump, fly, jump is considered different to jump, jump, fly. Let $u_0 = 1$.

a. Find the values of u_1, u_2, u_3. [3]

b. Show that $u_{n+2} = u_{n+1} + u_n$. [2]

c. (i) Write down the auxiliary equation for this recurrence relation. [5]

  (ii) Hence find the solution to this recurrence relation, giving your answer in the form $u_n = A\alpha^n + B\beta^n$ where α and β are to be determined exactly in surd form and $\alpha > \beta$. The constants A and B do not have to be found at this stage.

d. (i) Given that $A = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)$, use the value of u_0 to determine B. [3]

  (ii) Hence find the explicit formula for u_n.

e. Find the value of u_{20}. [1]

f. Find the smallest value of n for which $u_n > 100\,000$. [2]

Markscheme

a. $u_1 = 1$, $u_2 = 2$, $u_3 = 3$ **A1A1A1**

  [3 marks]

b. to get to step $n + 2$ she can either fly from step n or jump from step $n + 1$ **R2**

  so $u_{n+2} = u_{n+1} + u_n$ **AG**

  [2 marks]

c. (i) auxiliary equation $\lambda^2 - \lambda - 1 = 0$ **M1A1**

  Note: Award **M1** for attempting to write down a relevant quadratic.

  (ii) $\lambda = \frac{1 \pm \sqrt{1+4}}{2}$ **M1A1**

  $u_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$ **A1**

d. (i) $u_0 = 1$ implies $A + B = 1$. So $B = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$ *M1A1*

   (ii) $u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$ *A1*

Note: Accept equivalent expressions in parts (i) and (ii).

[3 marks]

e. $u_{20} = 10946$ *A1*

[1 mark]

f.   using table, smallest value for n is 25 (gives 121393)   *(M1)A1*

Note: Accept other methods, *eg*, logs on the dominant term.

[2 marks]

Examiners report

a.   [N/A]
b.   [N/A]
c.   [N/A]
d.   [N/A]
e.   [N/A]
f.   [N/A]

---

a.   Use the result $2003 = 6 \times 333 + 5$ and Fermat's little theorem to show that $2^{2003} \equiv 4(\bmod 7)$.   [3]

b.   Find $2^{2003}(\bmod 11)$ and $2^{2003}(\bmod 13)$.   [3]

c.   Use the Chinese remainder theorem, or otherwise, to evaluate $2^{2003}(\bmod 1001)$, noting that $1001 = 7 \times 11 \times 13$.   [7]

Markscheme

a. $2^{2003} = 2^5 \times (2^6)^{333}$ *M1A1*

   $\equiv 32 \times 1(\bmod 7)$ by Fermat's little theorem   *A1*
   $\equiv 4(\bmod 7)$ *AG*
   [3 marks]

b. $2003 = 3 + 10 \times 200$ *(M1)*

   $2^{2003} = 2^3 \times (2^{10})^{200} \, (\equiv 8 \times 1(\bmod 11)) \equiv 8(\bmod 11)$ *A1*
   $2^{2003} = 2^{11} \times (2^{12})^{166} \equiv 7(\bmod 13)$ *A1*
   [3 marks]

c.   form $M_1 = \frac{1001}{7} = 143$; $M_2 = \frac{1001}{11} = 91$; $M_3 = \frac{1001}{13} = 77$ *M1*

   solve $143x_1 \equiv 1(\bmod 7) \Rightarrow x_1 = 5$ *M1A1*
   $x_2 = 4$; $x_3 = 12$ *A1A1*
   $x = 4 \times 143 \times 5 + 8 \times 91 \times 4 + 7 \times 77 \times 12 = 12240 \equiv 228(\bmod 1001)$ *M1A1*

[7 marks]

Examiners report

a. Many candidates were able to complete part (a) and then went on to part (b). Some candidates raced through part (c). Others, who attempted part (c) using the alternative strategy of repeatedly solving linear congruencies, were sometimes successful.

b. Many candidates were able to complete part (a) and then went on to part (b). Some candidates raced through part (c). Others, who attempted part (c) using the alternative strategy of repeatedly solving linear congruencies, were sometimes successful.

c. Many candidates were able to complete part (a) and then went on to part (b). Some candidates raced through part (c). Others, who attempted part (c) using the alternative strategy of repeatedly solving linear congruencies, were sometimes successful.

Let the greatest common divisor of 861 and 957 be h .

a. Using the Euclidean algorithm, find h . [4]

b. Hence find integers A and B such that $861A + 957B = h$. [3]

c. Using part (b), solve $287w \equiv 2(\bmod 319)$, where $w \in \mathbb{N}, \ w < 319$. [5]

d. Find the general solution to the diophantine equation $861x + 957y = 6$. [6]

Markscheme

a. **METHOD 1**

by the above working on the left (or similar)   *M1A1A1*

Note: Award *A1* for 96 and *A1* for 93.

$h = 3$ (since 3 divides 93)   *A1*

[4 marks]

METHOD 2

$957 = 861 + 96$ *M1A1*

$861 = 8 \times 96 + 93$ *A1*

$96 = 93 + 3$

so $h = 3$ (since 3 divides 93)   *A1*

[4 marks]

b. **METHOD 1**

if method 1 was used for part (a)
by the above working on the right (or equivalent)   *M1*
$-10 \times 861 + 9 \times 957 = 3$
so $A = -10$ and $B = 9$ *A1A1*
[3 marks]
METHOD 2
$3 = 96 - 93$
$= 96 - (861 - 8 \times 96) = 9 \times 96 - 861$ *M1*
$= 9 \times (957 - 861) - 861$
$= -10 \times 861 + 9 \times 957$
so $A = -10$ and $B = 9$ *A1A1*
[3 marks]

c. from (b) $-10 \times 287 + 9 \times 319 = 1$ so   *A1*

$-10 \times 287 \equiv 1 (\bmod 319)$ *A1*
$287w \equiv 2 (\bmod 319) \Rightarrow -10 \times 287w \equiv -10 \times 2 (\bmod 319)$ *M1*
$\Rightarrow w \equiv -20 (\bmod 319)$ *A1*
so $w = 299$ *A1*
[5 marks]

d. from (b) $-10 \times 861 + 9 \times 957 = 3 \Rightarrow -20 \times 861 + 18 \times 957 = 6$ *M1A1*

so general solution is $x = -20 + 319t$ *A1A1*
$y = 18 - 287t \quad (t \in \mathbb{Z})$ *A1A1*
[6 marks]
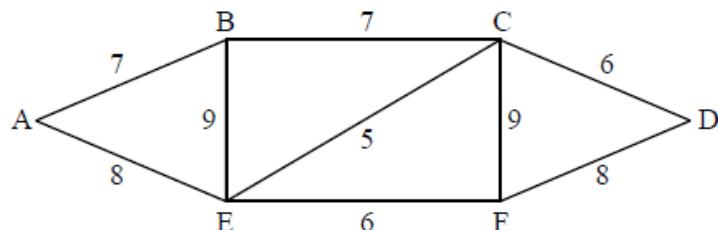
Examiners report

a. The Euclidean algorithm was well applied. If it is done in the format shown in the mark scheme then the keeping track method of the linear combinations of the 2 original numbers makes part (b) easier.

b. Again well answered but not quite as good as (a).

c. Surprisingly, since it is basic bookwork, this part was answered very badly indeed. Most candidates did not realise that -10 was the number to multiply by. Sadly, of the candidates that did do it, some did not read the question carefully enough to see that a positive integer answer was required.

d. Again this is standard bookwork. It was answered better than part (c). There were the usual mistakes in the final answer e.g. not having the two numbers, with the parameter, co-prime.

Consider the following weighted graph *G*.

a.i. State what feature of *G* ensures that *G* has an Eulerian trail. [1]

a.ii. State what feature of *G* ensures that *G* does not have an Eulerian circuit. [1]

b. Write down an Eulerian trail in *G*. [2]

c.i. State the Chinese postman problem. [2]

c.ii. Starting and finishing at B, find a solution to the Chinese postman problem for *G*. [3]

c.iii Calculate the total weight of the solution. [1]

Markscheme

a.i. *G* has an Eulerian trail because it has (exactly) two vertices (B and F) of odd degree   *R1*

   [1 mark]

a.ii. *G* does not have an Eulerian circuit because not all vertices are of even degree   *R1*

   [1 mark]

b. for example BAEBCEFCDF   *A1A1*

   Note: Award *A1* for start/finish at B/F, *A1* for the middle vertices.

   [2 marks]

c.i. to determine the shortest route (walk) around a weighted graph   *A1*

   using each edge (at least once, returning to the starting vertex)   *A1*

   Note: Correct terminology must be seen. Do not accept trail, path, cycle or circuit.

   [2 marks]

c.ii. we require the Eulerian trail in (b), (weight = 65)   *(M1)*

   and the minimum walk FEB (15)   *A1*

   for example BAEBCEFCDFEB   *A1*

   Note: Accept EB added to the end or FE added to the start of their answer in (b) in particular for follow through.

   [3 marks]

c.iii total weight is (65 + 15=)80   *A1*

   [1 mark]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]
c.i. [N/A]
c.ii. [N/A]
c.iii [N/A]

The simple, connected graph G has e edges and v vertices, where $v \geqslant 3$.

Given that both G and G' are planar and connected,

a. Show that the number of edges in G', the complement of G, is $\frac{1}{2}v^2 - \frac{1}{2}v - e$. [2]

b. show that the sum of the number of faces in G and the number of faces in G' is independent of e; [3]

c. show that $v^2 - 13v + 24 \leqslant 0$ and hence determine the maximum possible value of v. [7]

Markscheme

a. the total number of edges in G and G' is $\frac{v(v-1)}{2}$ **(A1)**

the number of edges in $G' = \frac{v(v-1)}{2} - e$ **A1**

$= \frac{1}{2}v^2 - \frac{1}{2}v - e$ **AG**

[2 marks]

b. using Euler's formula, number of faces in $G = e + 2 - v$ **A1**

number of faces in $G' = \frac{v^2}{2} - \frac{v}{2} - e + 2 - v$ **A1**

sum of these numbers $= \frac{v^2}{2} - \frac{5v}{2} + 4$ **A1**

this is independent of e **AG**

[3 marks]

c. for G to be planar, we require   **(M1)**

$e \leqslant 3v - 6$ **A1**

for $e \leqslant 3v - 6$ to be planar, we require

$\frac{v^2}{2} - \frac{v}{2} - e \leqslant 3v - 6$ **A1**

for these two inequalities to be satisfied simultaneously, adding or substituting we require

$\frac{v^2}{2} - \frac{v}{2} \leqslant 6v - 12$ **(M1)A1**

leading to $v^2 - 13v + 24 \leqslant 0$ **AG**

the roots of the equation are 10.8 (and 2.23)   **(A1)**

the largest value of v is therefore 10   **A1**

[7 marks]

Examiners report

a. Generally in this question, good candidates thought their way through it whereas weak candidates just wrote down anything they could off the

formula booklet or drew pictures of particular graphs. It was important to keep good notation and not let the same symbol stand for different

things.

(a) If they considered the complete graph they were fine.

b. Generally in this question, good candidates thought their way through it whereas weak candidates just wrote down anything they could off the formula booklet or drew pictures of particular graphs. It was important to keep good notation and not let the same symbol stand for different things.

(b) Some confusion here if they were not clear about which graph they were applying Euler's formula to. If they were methodical with good notation they obtained the answer.

c. Generally in this question, good candidates thought their way through it whereas weak candidates just wrote down anything they could off the formula booklet or drew pictures of particular graphs. It was important to keep good notation and not let the same symbol stand for different things.

(c) Again the same confusion about applying the inequality to both graphs. Most candidates realised which inequality was applicable. Many candidates had the good exam technique to pick up the last two marks even if they did not obtain the quadratic inequality.

The Fibonacci sequence can be described by the recurrence relation $f_{n+2} = f_{n+1} + f_n$ where $f_0 = 0$, $f_1 = 1$.

a. Write down the auxiliary equation and use it to find an expression for f_n in terms of n. [7]

b. It is known that $\alpha^2 = \alpha + 1$ where $\alpha = \frac{1+\sqrt{5}}{2}$. [8]

For integers $n \geq 3$, use strong induction on the recurrence relation $f_{n+2} = f_{n+1} + f_n$ to prove that $f_n > \alpha^{n-2}$.

Markscheme

a. attempt to find the auxiliary equation $(\lambda^2 - \lambda - 1 = 0)$ *M1*

$\lambda = \frac{1 \pm \sqrt{5}}{2}$ *(A1)*

the general solution is $f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$ *(M1)*

imposing initial conditions (substituting $n = 0,1$)   *M1*

$A + B = 0$ and $A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$ *A1*

$A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$ *A1*

$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ *A1*

Note: Condone use of decimal numbers rather than exact answers.

[7 marks]

b. let P(n) be $f_n > \alpha^{n-2}$ for integers $n \geq 3$

consideration of two consecutive values of f *R1*

$f_3 = 2$ and $\alpha^{3-2} = \frac{1+\sqrt{5}}{2}(1.618\ldots) \Rightarrow$ P (3) is true   *A1*

$f_4 = 3$ and $\alpha^{4-2} = \frac{3+\sqrt{5}}{2}(2.618\ldots) \Rightarrow$ P (4) is true   *A1*

Note: Do not award *A* marks for values of n other than $n = 3$ and $n = 4$.

(for $k \geq 4$), assume that P(k) and P($k - 1$) are true   *M1*

required to prove that P($k + 1$) is true

$f_{k+1} = f_k + f_{k-1}$ (and $f_k > \alpha^{k-2}$, $f_{k-1} > \alpha^{k-3}$)   **M1**

$f_{k+1} > \alpha^{k-2} + \alpha^{k-3} = \alpha^{k-3}(\alpha + 1)$ **A1**

$= \alpha^{k-3}\alpha^2 = \alpha^{k-1} = \alpha^{(k+1)-2}$ **A1**

as P(3) and P(4) are true, and P(k) , P($k-1$) true $\Rightarrow$ P($k+1$) true then P(k) is true for $k \geq 3$ by strong induction   **R1**

Note: To obtain the final **R1**, at least five of the previous marks must have been awarded.

[8 marks]

Examiners report

a. [N/A]

b. [N/A]

Use the Euclidean Algorithm to find the greatest common divisor of 7854 and 3315.

Hence state the number of solutions to the diophantine equation $7854x + 3315y = 41$ and justify your answer.

Markscheme

$7854 = 2 \times 3315 + 1224$ **M1A1**

$3315 = 2 \times 1224 + 867$ **A1**

$1224 = 1 \times 867 + 357$

$867 = 2 \times 357 + 153$

$357 = 2 \times 153 + 51$

$153 = 3 \times 51$ **A1**

The gcd is 51.   **A1**

Since 51 does not divide 41,   **R1**

there are no solutions.   **A1**

[7 marks]

Examiners report

Most candidates were able to use the Euclidean Algorithm correctly to find the greatest common divisor. Candidates who used the GCD button on their calculators were given no credit. Some candidates seemed unaware of the criterion for the solvability of Diophantine equations.

Consider the recurrence relation

$$u_n = 5u_{n-1} - 6u_{n-2}, \ u_0 = 0 \text{ and } u_1 = 1.$$

a. Find an expression for u_n in terms of n.   [6]

b. For every prime number $p > 3$, show that $p \mid u_{p-1}$. [4]

Markscheme

a. the auxiliary equation is $\lambda^2 - 5\lambda + 6 = 0$ *M1*

$\Rightarrow \lambda = 2, \ 3$ *(A1)*

the general solution is $u_n = A \times 2^n + B \times 3^n$ *A1*

imposing initial conditions (substituting $n = 0, \ 1$) *M1*

$A + B = 0$ and $2A + 3B = 1$ *A1*

the solution is $A = -1, \ B = 1$

so that $u_n = 3^n - 2^n$ *A1*

[6 marks]

b. $u_{p-1} = 3^{p-1} - 2^{p-1}$

$p > 3$, therefore 3 or 2 are not divisible by p *R1*

hence by FLT, $3^{p-1} \equiv 1 \equiv 2^{p-1} (\bmod\ p)$ for $p > 3$ *M1A1*

$u_{p-1} \equiv 0 (\bmod\ p)$ *A1*

$p \mid u_{p-1}$ for every prime number $p > 3$ *AG*

[4 marks]

Examiners report

a. [N/A]

b. [N/A]

The decimal number 1071 is equal to $a060$ in base b, where $a > 0$.

a. Convert the decimal number 1071 to base 12. [3]

b. Write the decimal number 1071 as a product of its prime factors. [1]

c.i. Using your answers to part (a) and (b), prove that there is only one possible value for b and state this value. [4]

c.ii. Hence state the value of a. [1]

Markscheme

a. **EITHER**

using a list of relevant powers of 12: 1, 12, 144 *(M1)*

$1071 = 7 \times 12^2 + 5 \times 12^1 + 3 \times 12^0$ *(A1)*

OR

attempted repeated division by 12 *(M1)*

$1071 \div 12 = 89\text{rem}3; \ 89 \div 12 = 7\text{rem}5$ *(A1)*

THEN

$1071 = 753_{12}$ *A1*

[3 marks]

b. $1071 = 3 \times 3 \times 7 \times 17$ *A1*

[1 mark]

c.i. in base b $a060$ ends in a zero and so b is a factor of 1071   *R1*

from part (a) $b < 12$ as $a060$ has four digits and so the possibilities are

$b = 3, \ b = 7$ **or** $b = 9$ *R1*

stating valid reasons to exclude both $b = 3$ *eg*, there is a digit of 6

and $b = 9$ *eg*, $1071 = (1420)_9$ *R1*

$b = 7$ *A1*

Note: The **A** mark is independent of the **R** marks.

[4 marks]

c.ii. $1071 = (3060)_7 \Rightarrow a = 3$ *A1*

[1 mark]

Examiners report

a. [N/A]
b. [N/A]
c.i. [N/A]
c.ii. [N/A]

---

a. Use the Euclidean algorithm to find the greatest common divisor of 264 and 1365.   [5]

b.i. Hence, or otherwise, find the general solution of the Diophantine equation   [6]

$$264x - 1365y = 3.$$

b.ii. Hence find the general solution of the Diophantine equation   [2]

$$264x - 1365y = 6.$$

c. By expressing each of 264 and 1365 as a product of its prime factors, determine the lowest common multiple of 264 and 1365.   [3]

Markscheme

a. $1365 = 5 \times 264 + 45$ *M1*

$264 = 5 \times 45 + 39$ *A1*

$45 = 1 \times 39 + 6$ **A1**

$39 = 6 \times 6 + 3$

$6 = 2 \times 3$ **A1**

so gcd is 3

[5 marks]

b.i. **EITHER**

$39 - 6 \times 6 = 3$ **(M1)**

$39 - 6 \times (45 - 39) = 3 \Rightarrow 7 \times 39 - 6 \times 45 = 3$ **(A1)**

$7 \times (264 - 5 \times 45) - 6 \times 45 = 3 \Rightarrow 7 \times 264 - 41 \times 45 = 3$ **(A1)**

$7 \times 264 - 41 \times (1365 - 5 \times 264) = 3 \Rightarrow 212 \times 264 - 41 \times 1365 = 3$ **(A1)**

OR

tracking the linear combinations when applying the Euclidean algorithm (could be displayed in (a))

$(1, 0)$	$(0, 1)$	
	$-5\,(0, 1)$	
	$(-5, 1)$	**(M1)**
$-5\,(-5, 1)$		
$(26, -5)$		**(A1)**
	$-(26, -5)$	
	$(-31, 6)$	**(A1)**
$-6\,(-31, 6)$		
$(212, -41)$		

$212 \times 264 - 41 \times 1365 = 3$ **(A1)**

THEN

a solution is $x = 212, y = 41$ (or equivalent *eg* $x = -243, y = -47$)   **(A1)**

$x = 212 + 455N, y = 41 + 88N$ (or equivalent) $(N \in \mathbb{Z})$ **A1**

[6 marks]

b.ii a solution is $x = 424, y = 82$ (or equivalent *eg* $x = -31,\ y = -6$)   **(A1)**

$x = 424 + 455N,\ y = 82 + 88N$ (or equivalent) $(N \in \mathbb{Z})$ **A1**

Note: Award **A1A0** for $x = 424 + 910N,\ y = 82 + 176N$.

[2 marks]

c. $264 = 2 \times 2 \times 2 \times 3 \times 11$ **A1**

$1365 = 3 \times 5 \times 7 \times 13$ **A1**

$1\,\mathrm{cm} = 2 \times 2 \times 2 \times 3 \times 5 \times 7 \times 11 \times 13 = 120120$ **A1**

Note: Only award marks if prime factorisation is used.

[3 marks]

Examiners report

---

b. Prove that a graph containing a triangle cannot be bipartite. [3]

c. Prove that the number of edges in a bipartite graph with n vertices is less than or equal to $\frac{n^2}{4}$. [5]

Markscheme

b. At least two of the three vertices in the triangle must lie on one of the two disjoint sets   *M1R1*

These two are joined by an edge so the graph cannot be bipartite   *R1*

[3 marks]

c. If there are x vertices in one of the two disjoint sets then there are $(n-x)$ vertices in the other disjoint set   *M1*

The greatest number of edges occurs when all vertices in one set are joined to all vertices in the other to give $x(n-x)$ edges   *A1*

Function $f(x) = x(n-x)$ has a parabolic graph.   *M1*

This graph has a unique maximum at $\left(\frac{n}{2}, \frac{n^2}{4}\right)$.   *A1*

so $x(n-x) \leqslant \frac{n^2}{4}$ *R1*

[5 marks]

Examiners report

b. Part (a) was usually done correctly but then clear argument for parts (b) and (c) were rare.

c. Part (a) was usually done correctly but then clear argument for parts (b) and (c) were rare.

The positive integer N is expressed in base p as $(a_n a_{n-1} \ldots a_1 a_0)_p$.

a. Show that when $p = 2$, N is even if and only if its least significant digit, a_0 , is 0. [5]

b. Show that when $p = 3$, N is even if and only if the sum of its digits is even. [6]

Markscheme

a. $N = a_n \times 2^n + a_{n-1} \times 2^{n-1} + \ldots + a_1 \times 2 + a_0$ *M1*

If $a_0 = 0$, then N is even because all the terms are even.   *R1*

Now consider

$a_0 = N - \sum_{r=1}^{n} a_r \times 2^r$ *M1*

If N is even, then a_0 is the difference of two even numbers and is therefore even.   *R1*

It must be zero since that is the only even digit in binary arithmetic.   *R1*

[5 marks]

b. $N = a_n \times 3^n + a_{n-1} \times 3^{n-1} + \ldots + a_1 \times 3 + a_0$

$= a_n \times (3^n - 1) + a_{n-1} \times (3^{n-1} - 1) + \ldots + a_1 \times (3 - 1) + a_n + a_{n-1} + \ldots + a_1 + a_0$ *M1A1*

Since 3^n is odd for all $n \in \mathbb{Z}^+$, it follows that $3^n - 1$ is even.   *R1*

Therefore if the sum of the digits is even, N is the sum of even numbers and is even.   *R1*

Now consider

$a_n + a_{n-1} + \ldots + a_1 + a_0 = N - \sum_{r=1}^{n} a_r(3^r - 1)$ *M1*

If N is even, then the sum of the digits is the difference of even numbers and is therefore even.   *R1*

[6 marks]

Examiners report

a. The response to this question was disappointing. Many candidates were successful in showing the 'if' parts of (a) and (b) but failed even to realise that they had to continue to prove the 'only if' parts.

b. The response to this question was disappointing. Many candidates were successful in showing the 'if' parts of (a) and (b) but failed even to realise that they had to continue to prove the 'only if' parts.

---

a. Given a sequence of non negative integers $\{a_r\}$ show that   [5]

(i) $\sum_{r=0}^{n} a_r(x+1)^r (\bmod\ x) \equiv \sum_{r=0}^{n} a_r(\bmod\ x)$ where $x \in \{2,\ 3,\ 4\ldots\}$.

(ii) $\sum_{r=0}^{n} (3a_{2r+1} + a_{2r})9^r = \sum_{r=0}^{2n+1} a_r 3^r$.

b. Hence determine whether the base 3 number 22010112200201 is divisible by 8.   [5]

Markscheme

a. (i) $(x+1)(\bmod\ x) \equiv 1(\bmod\ x)$ *(M1)*

$(x+1)^r(\bmod\ x)\ (\equiv 1^r(\bmod\ x)) = 1(\bmod\ x)$ *A1*

$\sum_{r=0}^{n} a_r(x+1)^r(\bmod\ x) \equiv \sum_{r=0}^{n} a_r(\bmod\ x)$ *AG*

(ii) **METHOD 1**

$\sum_{r=0}^{n} (3a_{2r+1} + a_{2r})9^r = \sum_{r=0}^{n} (3a_{2r+1} + a_{2r})3^{2r}$ *M1*

$= \sum_{r=0}^{n} (3^{2r+1}a_{2r+1} + 3^{2r}a_{2r})$ *M1A1*

$= \sum_{r=0}^{2n+1} a_r 3^r$ *AG*

METHOD 2

$$\sum_{r=0}^{n}(3a_{2r+1}+a_{2r})9^r = 3a_1 + a_0 + 9(3a_3 + a_2) + \ldots + 9^n(3a_{2n+1}+a_{2n}) \quad \textbf{\textit{A1}}$$

$$= a_0 + 3a_1 + 3^2a_2 + 3^3a_3 + \ldots + 3^{2n+1}a_{2n+1} \quad \textbf{\textit{M1A1}}$$

$$= \sum_{r=0}^{2n+1} a_r 3^r \quad \textbf{\textit{AG}}$$

[5 marks]

b. **METHOD 1**

using part (a) (ii) to separate the number into pairs of digits   *(M1)*

$22010112200201 \ (\text{base } 3) \equiv 8115621 \ (\text{base } 9)$ *A1*

using the sum of digits identity from part (a) (i)   *(M1)*

Note: Award *M1* if result from a(i) is used to show that the number is divisible by 2, even if no other valid working given.

$\text{sum} = 24$ *A1*

which is divisible by 8 *A1*

hence 22010112200201 (base 3) is divisible by 8

METHOD 2

$$\sum_{r=0}^{13} a_r 3^r \sum_{r=0}^{6}(3a_{2r+1}+a_{2r})9^r \quad \textbf{\textit{(M1)}}$$

$$\equiv \sum_{r=0}^{6}(3a_{2r+1}+a_{2r})(\bmod 8) \quad \textbf{\textit{A1}}$$

$$= a_0 + 3a_1 + a_2 + 3a_3 + \ldots + a_{12} + 3a_{13}(\bmod 8) \quad \textbf{\textit{M1}}$$

$$= (1 + 3 \times 0 + 2 + 3 \times 0 + \ldots + 3 \times 2)(\bmod 8) \equiv 24(\bmod 8) \quad \textbf{\textit{A1}}$$

$\equiv 0$ **OR** divisible by 8 *A1*

hence 22010112200201 (base 3) is divisible by 8

[5 marks]

Total [10 marks]

Examiners report

a. [N/A]

b. [N/A]

Consider the linear congruence $ax \equiv b \pmod{p}$ where $a,\ b,\ p,\ x \in \mathbb{Z}^+$, p is prime and a is not a multiple of p.

a. State Fermat's little theorem.   [2]

b.i. Use Fermat's little theorem to show that $x \equiv a^{p-2}b \pmod{p}$.   [3]

b.ii. Hence solve the linear congruence $5x \equiv 7 \pmod{13}$.   [3]

Markscheme

a. **EITHER**

if p is prime (and a is any integer) then $a^p \equiv a \pmod{p}$ **A1A1**

Note: Award **A1** for p prime and **A1** for the congruence or for stating that $p \mid a^p - a$.

OR

if p is prime (and $a \not\equiv 0 \pmod{p}$) then $a^{p-1} \equiv 1 \pmod{p}$ **A1A1**

Note: Award **A1** for p prime and **A1** for the congruence or for stating that $p \mid a^{p-1} - 1$.

Note: Condone use of equals sign provided $\pmod{p}$ is seen.

[2 marks]

b.i. multiplying both sides of the linear congruence by a^{p-2} **(M1)**

$a^{p-1}x \equiv a^{p-2}b \pmod{p}$ **A1**

as $a^{p-1} \equiv 1 \pmod{p}$ **R1**

$x \equiv a^{p-2}b \pmod{p}$ **AG**

[3 marks]

b.ii. $x \equiv 5^{11} \times 7 \pmod{13}$ **(M1)**

$\equiv 341796875 \pmod{13}$ **(A1)**

Note: Accept equivalent calculation *eg*, using $5^2 \equiv -1 \bmod 13$.

$\equiv 4 \pmod{13}$ **A1**

[3 marks]
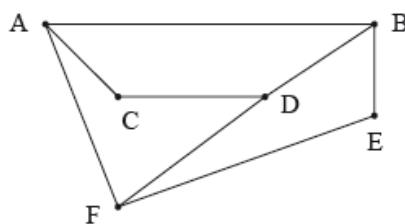
Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]

The following diagram shows the graph G.

a. Show that G is bipartite.   [2]

b. State which two vertices should be joined to make G equivalent to $K_{3,3}$.   [1]
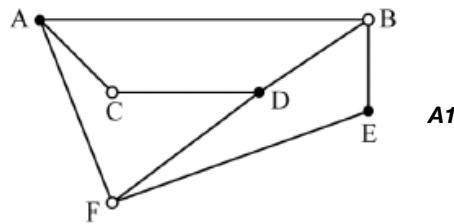
c. In a planar graph the degree of a face is defined as the number of edges adjacent to that face.   [2]

(i) Write down the degree of each of the four faces of G.

(ii) Explain why the sum of the degrees of all the faces is twice the number of edges.

d. H is a simple connected planar bipartite graph with e edges, f faces, v vertices and $v \geq 3$. [3]

Explain why there can be no face in H of degree

(i) one;

(ii) two;

(iii) three.

e. Hence prove that for H [3]

(i) $e \geq 2f$;

(ii) $e \leq 2v - 4$.

f. Hence prove that $K_{3,3}$ is not planar. [2]

Markscheme

a. shading alternate vertices or attempting to list pairs   **M1**

EITHER

A1

OR

ADE and BCF **A1**

as no two equal shadings are adjacent, the graph is bipartite   **AG**

[2 marks]

b. C and E **A1**

[1 mark]

c. (i) degree of each of the four faces is 4 **A1**

(ii) each edge bounds two faces   **R1**

[2 marks]

d. (i) simple so no loops   **R1**

(ii) simple so no multiple edges (and $v > 2$)   **R1**

(iii) bipartite graph so no triangles   **R1**

[3 marks]

e. (i) $2e = \sum \deg(f) \geq 4f$ **R1**

$e \geq 2f$ **AG**

(ii) if H is a simple connected planar graph then $e + 2 = v + f$ **M1**

$e + 2 - v \leq \frac{1}{2}e$ **M1**

$2e + 4 - 2v \leq e$

$e \leq 2v - 4$ **AG**

[3 marks]

f. for $K_{3,3}$ $v = 6$, and $e = 9$ **A1**

$9 \leq 2 \times 6 - 4$ is not true, therefore $K_{3,3}$ cannot be planar   **R1AG**

[2 marks]

Total [13 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]
f. [N/A]

---

a. Use the Euclidean algorithm to express gcd (123, 2347) in the form $123p + 2347q$, where $p, \ q \in \mathbb{Z}$.   [8]

b. Find the least positive solution of $123x \equiv 1(\bmod 2347)$.   [3]

c. Find the general solution of $123z \equiv 5(\bmod 2347)$.   [3]

d. State the solution set of $123y \equiv 1(\bmod 2346)$.   [1]

Markscheme

a. $2347 = 19 \times 123 + 10$ **M1A1**

$(123 = 12 \times 10 + 3)$
$10 = 3 \times 3 + 1$ **A1**
$1(\text{gcd}) = 10 - 3 \times 3 = 10 - 3 \times (123 - 12 \times 10)$ **M1A1**
$= 37 \times 10 - 3 \times 123$ **A1**
$= 37 \times (2347 - 19 \times 123) - 3 \times 123$ (for continuation)   **M1**
$= 37 \times 2347 - 706 \times 123$ **A1**
[8 marks]

b. **EITHER**

$1(\bmod 2347) = (-706 \times 123)(\bmod 2347)$ **M1A1**
OR
$x = -706 + 2347n$ **M1A1**
solution: 1641   **A1**
[3 marks]

c. $5(\bmod 2347) = (-3530 \times 123)(\bmod 2347)$ *(M1)*

GS : $z = -3530 + k2347$ **A1A1**
Note: Other common possibilities include $1164 + k2347$ and $8205 + k2347$.

[3 marks]

d. empty set (123 and 2346 both divisible by 3)   **A1**

[1 mark]

Examiners report

a. The majority of candidates were successful in parts (a) and (b). In part (c), some candidates failed to understand the distinction between a particular solution and a general solution. Part (d) was a 1 mark question that defeated all but the few who noticed that the gcd of the numbers concerned was 3.

b. The majority of candidates were successful in parts (a) and (b). In part (c), some candidates failed to understand the distinction between a particular solution and a general solution. Part (d) was a 1 mark question that defeated all but the few who noticed that the gcd of the numbers concerned was 3.

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d. The majority of candidates were successful in parts (a) and (b). In part (c), some candidates failed to understand the distinction between a particular solution and a general solution. Part (d) was a 1 mark question that defeated all but the few who noticed that the gcd of the numbers concerned was 3.

The planar graph G and its complement G' are both simple and connected.

Given that G has 6 vertices and 10 edges, show that G' is a tree.

Markscheme

the complete graph with 6 vertices has 15 edges so G' has

6 vertices and 5 edges   *M1A1*

the number of faces in G' , $f = 2 + e - v = 1$ *M1A1*

it is therefore a tree because $f = 1$ *R1*

Note: Accept it is a tree because $v = e + 1$

[5 marks]

Examiners report
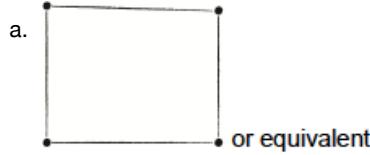
Part (a) was well answered by many candidates.

The graph $K_{2,2}$ is the complete bipartite graph whose vertex set is the disjoint union of two subsets each of order two.

a. Draw $K_{2,2}$ as a planar graph.   [2]

b. Draw a spanning tree for $K_{2,2}$.   [1]

c. Draw the graph of the complement of $K_{2,\,2}$. [1]

d. Show that the complement of any complete bipartite graph does not possess a spanning tree. [3]

Markscheme

a.

A1A1

or equivalent

Note: Award **A1** for a correct version of $K_{2,\,2}$ and **A1** for a correct planar representation.

[2 marks]

b.

A1

[1 mark]

c.

A1

[1 mark]

d. the complete bipartite graph $K_{m,\,n}$ has two subsets of vertices A and B, such that each element of A is connected to every element of B **A1**

in the complement, no element of A is connected to any element of B. The complement is not a connected graph   **A1**

by definition a tree is connected   **R1**

hence the complement of any complete bipartite graph does not possess a spanning tree   **AG**

[3 marks]

Total [7 marks]

Examiners report

a. Part (a) was generally well done with a large number of candidates drawing a correct planar representation for $K_{2,2}$. Some candidates, however, produced a correct non-planar representation of $K_{2,2}$.

b. Parts (b) and (c) were generally well done with many candidates drawing a correct spanning tree for $K_{2,2}$ and the correct complement of $K_{2,2}$.

c. Parts (b) and (c) were generally well done with many candidates drawing a correct spanning tree for $K_{2,2}$ and the correct complement of $K_{2,2}$.

d. Part (d) tested a candidate's ability to produce a reasoned argument that clearly explained why the complement of $K_{m,n}$ does not possess a spanning tree. This was a question part in which only the best candidates provided the necessary rigour in explanation.

Throughout this question, $(abc\ldots)_n$ denotes the number $abc\ldots$ written with number base n. For example $(359)_n = 3n^2 + 5n + 9$.

a. (i)   Given that $(43)_n \times (56)_n = (3112)_n$, show that $3n^3 - 19n^2 - 38n - 16 = 0$. [3]

   (ii)   Hence determine the value of n.

b. Determine the set of values of n satisfying $(13)_n \times (21)_n = (273)_n$. [3]

c. Show that there are no possible values of n satisfying $(32)_n \times (61)_n = (1839)_n$. [4]

Markscheme

a. (i) n satisfies the equation $(4n + 3)(5n + 6) = 3n^3 + n^2 + n + 2$ *M1A1*

   $3n^3 - 19n^2 - 38n - 16 = 0$ *(AG)*

   (ii) $n = 8$ *A1*

   Note: If extra solutions $(-1, \ -2/3)$ are not rejected (them just not appearing is fine) do not award the final *A1*.

   [3 marks]

b. n satisfies the equation $(n + 3)(2n + 1) = 2n^2 + 7n + 3$ *A1*

   this is an identity satisfied by all n *(A1)*

   $n > 7$ or $n \geqslant 8$ *A1*

   [3 marks]

c. n satisfies the equation $(3n + 2)(6n + 1) = n^3 + 8n^2 + 3n + 9$ *A1*

   $n^3 - 10n^2 - 12n + 7 = 0$ *A1*

   roots are 11.03, 0.434 and –1.46   *A1*

   since there are no integer roots therefore the product is not true in any number base   *R1AG*

   Note: Accept an argument by contradiction that considers the equation modulo n, with $n > 10$.

   [4 marks]

Examiners report

a. Well answered.

b. The fact that this gave an identity was managed by most. Then some showed their misunderstanding by saying any real number. Few noticed that the digit 7 means that the base must be greater than 7.

c. The cubic equation was generally reached but many candidates then forgot what type of number n had to be. To justify that there are no positive integer roots you need to write down what the roots are. There were a couple of really neat solutions that obtained a contradiction by working modulo n.

---

a. Solve the recurrence relation $v_n + 4v_{n-1} + 4v_{n-2} = 0$ where $v_1 = 0, \ v_2 = 1$. [6]

b. Use strong induction to prove that the solution to the recurrence relation $u_n - 4u_{n-1} + 4u_{n-2} = 0$ where $u_1 = 0$, $u_2 = 1$ is given by [8]

$u_n = 2^{n-2}(n-1)$.

c. Find a simplified expression for $u_n + v_n$ given that, [3]

(i) n is even.

(ii) n is odd.

Markscheme

a. the auxiliary equation is $m^2 + 4m + 4 = 0$ **M1**

$m = -2$ **A1**

the general solution is

$v_n = (A + Bn) \times (-2)^n$ **A1**

the boundary conditions give

$0 = -2(A + B)$

$1 = 4(A + 2B)$ **M1**

the solution is $A = -\frac{1}{4}$, $B = \frac{1}{4}$ **A1A1**

so that $v_n = \frac{1}{4}(n-1)(-2)^n \ \left(\text{or } (n-1)(-2)^{n-2}\right)$

[6 marks]

b. $n = 1$ gives $(1 - 1) \times \frac{1}{2} = 0$ which is correct **A1**

$n = 2$ gives $(2 - 1) \times 1 = 1$ which is correct **A1**

Note: Must be checked for $n = 1$ and 2, other values gain no marks.

assume that the result is true for all positive integers $\leqslant k$ **M1**

$u_{k+1} = 4u_k - 4u_{k-1}$ **(M1)**

$u_{k+1} = 4 \times 2^{k-2}(k-1) - 4 \times 2^{k-3}(k-2)$ **A1**

$= 2^{k-1}(2k - 2 - k + 2)$ or equivalent **A1**

$= k2^{k-1}$ **A1**

therefore true for $n \leqslant k \Rightarrow$ true for $n = k + 1$ and since true for $n = 1$ and $n = 2$, the result is proved by strong induction **R1**

Note: Only award the **R1** if at least four of the above marks have been awarded.

Note: Allow true for k and $k - 1$ (in 2 places) instead of stronger statement.

Note: First **M1** does not have to be given for further marks to be gained but second **(M1)** does.

[8 marks]

c. (i) $u_n + v_n = 2^{n-2}(n-1) + (-2)^{n-2}(n-1)$

when n is even $u_n + v_n = 2^{n-2}(n-1) + 2^{n-2}(n-1)$ **M1**

$= 2^{n-1}(n-1)$ **A1**

(ii) when n is odd $u_n + v_n = 0$ **A1**

[3 marks]

Examiners report

a. This was either done well and completely correct or very little achieved at all (working out v_0 for some reason). As expected a few candidates forgot what to do for a repeated root. The varied response to this question was surprising since it is just standard book work.

b. Strong induction proved to be a very good discriminator. Some candidates knew exactly what to do and did it well, others had no idea. Common mistakes were not checking $n = 2$ and 2, trying ordinary induction and worse of all assuming the very thing that they were trying to prove.

c. Most candidates that had the 2 expressions, knew how to get rid of the minus sign in the 2 cases. Some candidates could not attempt this as they had not completed part (a) although when it was wrong, follow through marks could be obtained.

---

a. Use the Euclidean algorithm to show that 1463 and 389 are relatively prime. [4]

b. Find positive integers a and b such that [5]

$$1463a - 389b = 1$$

.

Markscheme

a. $1463 = 3 \times 389 + 296$ **M1A1**

$389 = 1 \times 296 + 93$

$296 = 3 \times 93 + 17$ **A1**

$93 = 5 \times 17 + 8$

$17 = 2 \times 8 + 1$ which shows that the gcd is 1   **A1**

hence 1463 and 389 are relatively prime   **AG**

[4 marks]

b. **EITHER**

$1 = 17 - 2 \times 8$ **(M1)**

$= 17 - 2 \times (93 - 5 \times 17) = 11 \times 17 - 2 \times 93$ **(A1)**

$= 11 \times (296 - 3 \times 93) - 2 \times 93 = 11 \times 296 - 35 \times 93$ **(A1)**

$= 11 \times 296 - 35 \times (389 - 296) = 46 \times 296 - 35 \times 389$

$= 46 \times (1463 - 3 \times 389) - 35 \times 389$ **(A1)**

$= 46 \times 1463 - 173 \times 389$ **A1**

$(a = 46, \ b = 173)$

OR

method of keeping track of the linear combinations from the beginning (could be seen alongside the working in (a))

$(1, 0) \ (0, 1)$
$\quad \quad -3(1, 0)$
$\quad \quad (-3, 1)$
$\quad$ **(M1)(A1)**

$-(-3, 1)$
$(4, \ -1)$

$$-3(4, \ -1)$$
$$(-15, \ 4)$$

(A1)

$$-5(-15, \ 4)$$
$$(79, \ -21) \quad \text{\textit{(A1)}}$$

$$-2(79, \ -21)$$
$$(-173, \ 46)$$

so $-173 \times 389 + 46 \times 1463 = 1$ giving $46 \times 1463 - 173 \times 389 = 1$ **A1**

$$(a = 46, \ b = 173)$$

Note: Accept any positive answers of the form $a = 46 + 389t$, $b = 173 + 1463t$, t an integer.

[5 marks]

Examiners report

a. Very well answered. Some candidates lost the final mark by not saying that their working showed that the greatest common divisor (gcd) was 1.

b. The working backwards method was generally well known but there were arithmetic mistakes. Some candidates did not realise that their aim was to keep 1 as a combination of *two* remainders. The final answer could have been checked with the calculator as could intermediary steps. What was sadly less well known was the linear combinations format of laying the work out. See the OR method in the mark scheme. This makes the numerical work much less tedious and deserves to be better known.

a.i. In the context of graph theory, explain briefly what is meant by a circuit;   [1]

a.ii. In the context of graph theory, explain briefly what is meant by an Eulerian circuit.   [1]

b. The graph G has six vertices and an Eulerian circuit. Determine whether or not its complement G' can have an Eulerian circuit.   [3]

c. Find an example of a graph H, with five vertices, such that H and its complement H' both have an Eulerian trail but neither has an Eulerian   [4]

circuit. Draw H and H' as your solution.

Markscheme

a.i. a circuit is a walk that begins and ends at the same vertex and has no repeated edges   **A1**

[1 mark]

a.ii. an Eulerian circuit is a circuit that contains every edge of a graph   **A1**

[1 mark]

b. if G has an Eulerian circuit all vertices are even (are of degree 2 or 4)   **A1**
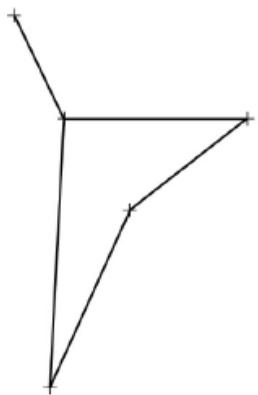
hence, G' must have all vertices odd (of degree 1 or 3)   **R1**

hence, G' cannot have an Eulerian circuit   **R1**
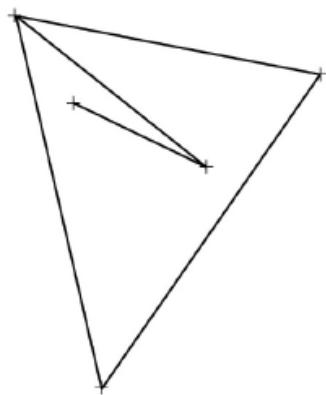
Note: Award **A1** to candidates who begin by considering a specific G and G' (diagram). Award **R1R1** to candidates who then consider a general G and G'.

[3 marks]

c. for example

A2

A2

Notes: Each graph must have 3 vertices of order 2 and 2 odd vertices. Award **A2** if one of the graphs satisfies that and the final **A2** if the other graph is its complement.

[4 marks]

Examiners report

When numbers are written in base n, $33^2 = 1331$.

a. By writing down an appropriate polynomial equation, determine the value of n. [4]

b. Rewrite the above equation with numbers in base 7. [6]

Markscheme

a. the equation can be written as

$(3n + 3)^2 = n^3 + 3n^2 + 3n + 1$ **M1A1**

any valid method of solution giving $n = 8$ **(M1)A1**

Note: Attempt to change at least one side into an equation in n gains the **M1**.

[4 marks]

b. **METHOD 1**

as decimal numbers,

$(33)_8 = 27, \; (1331)_8 = 729$ *A1A1*

converting to base 7 numbers,

$\quad 27 = (36)_7$ *A1*
$\quad 7)\underline{729}$ *M1*
$\quad 7)\underline{104}(1$
$\quad 7)\underline{\,14}(6$
$\quad 7)\underline{\,\,2}(0$
$\quad 7)\underline{\,\,0}(2$

therefore $729 = (2061)_7$ *A1*

the required equation is

$36^2 = 2061$ *A1*

METHOD 2

as a decimal number, $(33)_8 = 27$ *A1*

converting to base 7,

$27 = (36)_7$ *A1*

multiplying base 7 numbers
$\quad\quad 36$
$\quad\underline{\times\, 36}$
$\quad 1440$ *M1*
$\quad\underline{\;\;321}$ *A1*
$\quad 2061$ *A1*

the required equation is

$36^2 = 2061$ *A1*

Note: Allow *M1* for showing the method of converting a number to base 7 regardless of what number they convert.

[6 marks]

Examiners report

a. Part (a) was a good indicator of overall ability. Many candidates did not write both sides of the equation in terms of n and thus had an impossible equation, which should have made them realise that they had a mistake. The answers that were given in (a) and (b) could have been checked, so that the candidate knew they had done it correctly.

b. Part (b) was not well answered and of those candidates that did, some only gave one side of the equation in base 7. The answers that were given in (a) and (b) could have been checked, so that the candidate knew they had done it correctly.

Let $f(n) = n^5 - n$, $n \in \mathbb{Z}^+$.

a. Find the values of $f(3)$, $f(4)$ and $f(5)$. [2]

b. Use the Euclidean algorithm to find [4]

    (i) $\gcd(f(3),\ f(4))$;

    (ii) $\gcd(f(4),\ f(5))$.

c. Explain why $f(n)$ is always exactly divisible by 5. [1]

d. By factorizing $f(n)$ explain why it is always exactly divisible by 6. [4]

e. Determine the values of n for which $f(n)$ is exactly divisible by 60. [3]

Markscheme

a. $240, 1020, 3120$ ***A2***

Note: Award ***A2*** for three correct answers, ***A1*** for two correct answers.

[2 marks]

b. (i) $1020 = 240 \times 4 + 60$ ***(M1)***

$240 = 60 \times 4$

$\gcd(1020,\ 240) = 60$ ***A1***

(ii) $3120 = 1020 \times 3 + 60$ ***(M1)***

$1020 = 60 \times 17$

$\gcd(1020,\ 3120) = 60$ ***A1***

Note: Must be done by Euclid's algorithm.

[4 marks]

c. by Fermat's little theorem with $p = 5$ ***R1***

$n^5 \equiv n(\bmod 5)$

so 5 divides $f(n)$

[1 mark]

d. $f(n) = n(n^2 - 1)(n^2 + 1) = n(n - 1)(n + 1)(n^2 + 1)$ ***(A1)A1***

$n - 1,\ n,\ n + 1$ are consecutive integers and so contain a multiple of 2 and 3 ***R1R1***

Note: Award ***R1*** for justification of 2 and ***R1*** for justification of 3.

And therefore $f(n)$ is a multiple of 6.   ***AG***

[4 marks]

e. from (c) and (d) $f(n)$ is always divisible by 30   **R1**

considering the factorization, it is divisible by 60 when n is an odd number   **A1**

or when n is a multiple of 4   **A1**

Note: Accept answers such as when n is not congruent to $2(\bmod 4)$.

[3 marks]

Total [14 marks]

Examiners report

a. Well answered.

b. Also well answered. A few candidates did not use the Euclidean algorithm to find the gcd as instructed.

c. Many candidates essential said it was true because it was! There is only one mark which means one minute, so it must be a short answer which it

is by using Fermat's Little Theorem.

d. Some good answers but too many did not factorize as instructed, so that they could then spot the consecutive numbers.

e. Only the better candidates realised that they had to find another factor of 2.

a. (i)   One version of Fermat's little theorem states that, under certain conditions,   [8]

$$a^{p-1} \equiv 1(\bmod p).$$

Show that this result is not valid when $a = 4, p = 9$ and state which
condition is not satisfied.

(ii)   Given that $5^{64} \equiv n(\bmod 7)$, where $0 \leqslant n \leqslant 6$, find the value of n.

b. Find the general solution to the simultaneous congruences   [6]

$$x \equiv 3(\bmod 4)$$

$$3x \equiv 2(\bmod 5).$$

Markscheme

a. (i) $4^8 = 65536 \equiv 7(\bmod 9)$ *A1*

not valid because 9 is not a prime number   *R1*

Note: The *R1* is independent of the *A1*.

(ii)   using Fermat's little theorem   *M1*

$5^6 \equiv 1(\bmod 7)$ *A1*

therefore

$(5^6)^{10} = 5^{60} \equiv 1(\bmod 7)$ ***A1***

also, $5^4 = 625$ ***M1***

$\equiv 2(\bmod 7)$ ***A1***

therefore

$5^{64} \equiv 1 \times 2 \equiv 2(\bmod 7)$ (so $n = 2$)   ***A1***

Note: Accept alternative solutions not using Fermat.

[8 marks]

b. **EITHER**

solutions to $x \equiv 3(\bmod 4)$ are

3, 7, 11, 15, 19, 23, 27, …   ***A1***

solutions to $3x \equiv 2(\bmod 5)$. are

4, 9, 14, 19 …   ***(M1)A1***

so a solution is $x = 19$ ***A1***

using the Chinese remainder theorem (or otherwise)   ***(M1)***

the general solution is $x = 19 + 20n \ (n \in \mathbb{Z})$ ***A1***

(accept 19(mod 20))

OR

$x = 3 + 4t \Rightarrow 9 + 12t \equiv 2(\bmod 5)$ ***M1A1***

$\Rightarrow 2t \equiv 3(\bmod 5)$ ***A1***

$\Rightarrow 6t \equiv 9(\bmod 5)$

$\Rightarrow t \equiv 4(\bmod 5)$ ***A1***

so $t = 4 + 5n$ and $x = 19 + 20n \ (n \in \mathbb{Z})$ ***M1A1***

(accept 19(mod 20))

Note: Also accept solutions done by formula.

[6 marks]

Examiners report

a. Part (a) was generally well answered with a variety of methods seen in (a)(ii). This was set with Fermat's Little Theorem in mind but in the event many candidates started off with many different powers of 5, eg $5^4 \equiv 2$, $5^8 \equiv 4$ and $5^3 \equiv -1(\bmod 7)$ were all seen. A variety of methods was also seen in (b), ranging from use of the Chinese Remainder Theorem, finding tables of numbers congruent to 3(mod 4)and 4(mod 5) and the use of an appropriate formula.

b. Part (a) was generally well answered with a variety of methods seen in (a)(ii). This was set with Fermat's Little Theorem in mind but in the event many candidates started off with many different powers of 5, eg $5^4 \equiv 2$, $5^8 \equiv 4$ and $5^3 \equiv -1(\bmod 7)$ were all seen. A variety of methods was also seen in (b), ranging from use of the Chinese Remainder Theorem, finding tables of numbers congruent to 3(mod 4)and

4(mod 5) and the use of an appropriate formula.

(a) (i) Write down the general solution of the recurrence relation $u_n + 2u_{n-1} = 0$, $n \geqslant 1$.

(ii) Find a particular solution of the recurrence relation $u_n + 2u_{n-1} = 3n - 2$, $n \geqslant 1$, in the form $u_n = An + B$, where $A,\ B \in \mathbb{Z}$.

(iii) Hence, find the solution to $u_n + 2u_{n-1} = 3n - 2$, $n \geqslant 1$ where $u_1 = 7$.

(b) Find the solution of the recurrence relation $u_n = 2u_{n-1} - 2u_{n-2}$, $n \geqslant 2$, where $u_0 = 2$, $u_1 = 2$. Express your solution in the form $2^{f(n)} \cos(g(n)\pi)$, where the functions f and g map $\mathbb{N}$ to $\mathbb{R}$.

Markscheme

(a) (i) use of auxiliary equation or recognition of a geometric sequence **(M1)**

$u_n = (-2)^n u_0$ or $= \mathrm{A}(-2)^n$ or $u_1(-2)^{n-1}$ **A1**

(ii) substitute suggested solution **M1**

$An + B + 2\left(A(n-1) + B\right) = 3n - 2$ **A1**

equate coefficients of powers of n and attempt to solve **(M1)**

$A = 1,\ B = 0$ **A1**

(so particular solution is $u_n = n$)

(iii) use of general solution = particular solution + homogeneous solution **(M1)**

$u_n = \mathrm{C}(-2)^2 + n$ **A1**

attempt to find C using $u_1 = 7$ **M1**

$u_n = -3(-2)^n + n$ **A1**

[10 marks]

(b) the auxiliary equation is $r^2 - 2r + 2 = 0$ **A1**

solutions: $r_1,\ r_2 = 1 \pm \mathrm{i}$ **A1**

general solution of the recurrence: $u_n = \mathrm{A}(1 + \mathrm{i})^n + \mathrm{B}(1 - \mathrm{i})^n$ or trig form **A1**

attempt to impose initial conditions **M1**

$A = B = 1$ or corresponding constants for trig form **A1**

$u_n = 2^{\left(\frac{n}{2}+1\right)} \times \cos\left(\frac{n\pi}{4}\right)$ **A1A1**

[7 marks]

Total [17 marks]

Examiners report

[N/A]

A version of Fermat's little theorem states that when p is prime, a is a positive integer and a and p are relatively prime, then $a^{p-1} \equiv 1(\bmod\ p)$.

a. Use the above result to show that if p is prime then $a^p \equiv a(\bmod\ p)$ where a is any positive integer. [4]

b. Show that $2^{341} \equiv 2(\bmod\ 341)$. [7]

c. (i) State the converse of the result in part (a). [2]

  (ii) Show that this converse is not true.

Markscheme

a. consider two cases **M1**

  let a and p be coprime

  $a^{p-1} \equiv 1(\bmod\ p)$ **R1**

  $\Rightarrow a^p \equiv a(\bmod\ p)$

  let a and p not be coprime

  $a \equiv 0(\bmod\ p)$ **M1**

  $a^p = 0(\bmod\ p)$ **R1**

  $\Rightarrow a^p = a(\bmod\ p)$

  so $a^p = a(\bmod\ p)$ in both cases **AG**

  [4 marks]

b. $341 = 11 \times 31$ **(M1)**

  we know by Fermat's little theorem

  $2^{10} \equiv 1(\bmod\ 11)$ **M1**

  $\Rightarrow 2^{341} \equiv (2^{10})^{34} \times 2 \equiv 1^{34} \times 2 \equiv 2(\bmod\ 11)$ **A1**

  also $2^{30} \equiv 1(\bmod\ 31)$ **M1**

  $\Rightarrow 2^{341} \equiv (2^{30})^{11} \times 2^{11}$ **A1**

  $\equiv 1^{11} \times 2048 \equiv 2(\bmod\ 31)$ **A1**

  since 31 and 11 are coprime **R1**

  $2^{341} \equiv 2(\bmod\ 341)$ **AG**

  [7 marks]

c. (i) converse: if $a^p = a(\bmod\ p)$ then p is a prime **A1**

  (ii) from part (b) we know $2^{341} \equiv 2(\bmod\ 341)$

  however, 341 is composite

  hence 341 is a counter-example and the converse is not true **R1**

  [2 marks]

Examiners report

a. There were very few fully correct answers. In (b) the majority of candidates assumed that 341 is a prime number and in (c) only a handful of

  candidates were able to state the converse.

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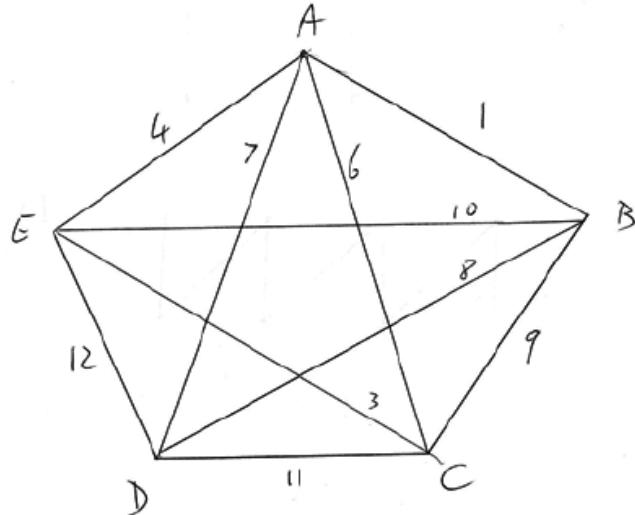
The weighted graph K, representing the travelling costs between five customers, has the following adjacency table.

	A	B	C	D	E
A	0	1	6	7	4
B	1	0	9	8	10
C	6	9	0	11	3
D	7	8	11	0	12
E	4	10	3	12	0

a. Draw the graph K. [2]

b. Starting from customer D, use the nearest-neighbour algorithm, to determine an upper bound to the travelling salesman problem for K. [4]

c. By removing customer A, use the method of vertex deletion, to determine a lower bound to the travelling salesman problem for K. [4]

Markscheme

a.

complete graph on five vertices **A1**
weights correctly marked on graph **A1**
[2 marks]

b. clear indication that the nearest-neighbour algorithm has been applied **M1**

DA (or 7) **A1**
AB (or 1) BC (or 9) **A1**
CE (or 3), ED (or 12), giving UB = 32 **A1**

[4 marks]

c. attempt to use the vertex deletion method   *M1*

minimum spanning tree is ECBD   *A1*
(EC 3, BD 8, BC 9 total 20)
reconnect A with the two edges of least weight, namely AB (1) and AE (4)   *M1*
lower bound is 25   *A1*
[4 marks]

Examiners report

---

a. (i)   Given that $a \equiv d(\bmod\ n)$ and $b \equiv c(\bmod\ n)$ prove that   [11]

$$(a + b) \equiv (c + d)(\bmod\ n) .$$

(ii)   Hence solve the system

$$\begin{cases} 2x + 5y \equiv 1(\bmod\ 6) \\ x + y \equiv 5(\bmod\ 6) \end{cases}$$

b. Show that $x^{97} - x + 1 \equiv 0(\bmod\ 97)$ has no solution.   [3]

Markscheme

a. (i) $a \equiv d(\bmod\ n)$ and $b \equiv c(\bmod\ n)$

so $a - d = pn$ and $b - c = qn$,   *M1A1*
$a - d + b - c = pn + qn$
$(a + b) - (c + d) = n(p + q)$ *A1*
$(a + b) \equiv (c + d)(\bmod\ n)$ *AG*

(ii) $\begin{cases} 2x + 5y \equiv 1(\bmod\ 6) \\ x + y \equiv 5(\bmod\ 6) \end{cases}$

adding $3x + 6y \equiv 0(\bmod\ 6)$ *M1*

$6y \equiv 0(\bmod\ 6)$ so $3x \equiv 0(\bmod\ 6)$ *R1*

$x \equiv 0$ or $x \equiv 2$ or $x \equiv 4(\bmod\ 6)$ *A1A1A1*

for $x \equiv 0,\ 0 + y \equiv 5(\bmod\ 6)$ so $y \equiv 5(\bmod\ 6)$ *A1*

for $x \equiv 2,\ 2 + y \equiv 5(\bmod\ 6)$ so $y \equiv 3(\bmod\ 6)$ *A1*

If $x \equiv 4(\bmod\ 6),\ 4 + y \equiv 5(\bmod\ 6)$ so $y \equiv 1(\bmod\ 6)$ *A1*

[11 marks]

b. Suppose x is a solution

97 is prime so $x^{97} \equiv x(\bmod\ 97)$ *M1*
$x^{97} - x \equiv 0(\bmod\ 97)$ *A1*
$x^{97} - x + 1 \equiv 1 \neq 0(\bmod\ 97)$
Hence there are no solutions   *R1*

[3 marks]

Examiners report

a. Part (a) (i) was not found difficult but using it in part (a)(ii) resulted in two or three correct lines and then abandonment of the problem.

b. Part (a) (i) was not found difficult but using it in part (a)(ii) resulted in two or three correct lines and then abandonment of the problem.

---

a. The sequence $\{u_n\}$, $n \in \mathbb{N}$, satisfies the recurrence relation $u_{n+1} = 7u_n - 6$. [5]

Given that $u_0 = 5$, find an expression for u_n in terms of n.

b. The sequence $\{v_n\}$, $n \in \mathbb{N}$, satisfies the recurrence relation $v_{n+2} = 10v_{n+1} + 11v_n$. [7]

Given that $v_0 = 4$ and $v_1 = 44$, find an expression for v_n in terms of n.

c. The sequence $\{v_n\}$, $n \in \mathbb{N}$, satisfies the recurrence relation $v_{n+2} = 10v_{n+1} + 11v_n$. [4]

Show that $v_n - u_n \equiv 15(\bmod\ 16)$, $n \in \mathbb{N}$.

Markscheme

a. **METHOD 1**

attempting to find a solution in the form $u_n = A7^n + B$ *M1*

EITHER

eg, and $u_0 = 5 \Rightarrow 5 = A + B$ and $u_1 = 29 \Rightarrow 29 = 7A + B$ *A1*

OR

$A7^{n+1} + B = A7^{n+1} + 7B - 6$ (or equivalent)   *A1*

THEN

attempting to solve for A and B *(M1)*

$u_n = 4 \times 7^n + 1$ *A1A1*

Note: Accept $A = 4$, $B = 1$ provided the first *M1* is awarded.

METHOD 2

attempting an iterative method *eg*, $u_1 = 7(5) - 6$ and

$u_2 = 7^2(5) - 6(7 + 1)\ (etc)$ *(M1)*

$u_n = 5 \times 7^n - 6\left(\frac{7^n - 1}{7 - 1}\right)$ *M1A1*

Note: Award *M1* for attempting to express u_n in terms of n.

$u_n = 4 \times 7^n + 1$ *A1A1*

METHOD 3

attempting to find a solution in the form $u_n = A7^n + B$ **M1**

$A(n + 1) + B = 7(An + B) - 6$

$7B - 6 = B$ **A1**

attempting to solve for A **(M1)**

$u_n = 4 \times 7^n + 1$ **A1A1**

METHOD 4

$u_{n+1} - 7u_n + 6 - (u_n - 7u_{n+1} + 6) = 0 \Rightarrow u_{n+1} - 8u_n + 7u_{n-1} = 0$

$r^2 - 8r + 7 = 0$

$r = 1, \ 7$

attempting to find a solution in the form $u_n = A7^n + B$ **M1**

EITHER

eg, $u_0 = 5 \Rightarrow 5 = A + B$ and $u_1 = 29 \Rightarrow 29 = 7A + B$ **A1**

OR

$A7^{n+1} + B = A7^{n+1} + 7B - 6$ (or equivalent) **A1**

THEN

attempting to solve for A and B **(M1)**

$u_n = 4 \times 7^n + 1$ **A1A1**

[5 marks]

b. attempting to find the auxiliary equation **M1**

$r^2 - 10r - 11 = 0$ $((r - 11)(r + 1) = 0)$ **A1**

$r = 11, \ r = -1$ **A1**

$v_n = A11^n + B(-1)^n$ **(M1)**

attempting to use the initial conditions **M1**

$A + B = 4, \ 11A - B = 44$ **A1**

$v_n = 4 \times 11^n$ **A1**

[7 marks]

c. $v_n - u_n = 4(11^n - 7^n) - 1$ **M1**

EITHER

$= 4(11 - 7)(11^{n-1} + \ldots + 7^{n-1}) - 1$ **M1A1**

OR

$= 4\left((7 + 4)^n - 7^n\right) - 1$ **A1**

subtracting the 7^n from the expanded first bracket **M1**

THEN

obtaining 16 times a whole number -1 **A1**

$v_n - u_n \equiv 15(\bmod 16), \ n \in \mathbb{N}$ **AG**
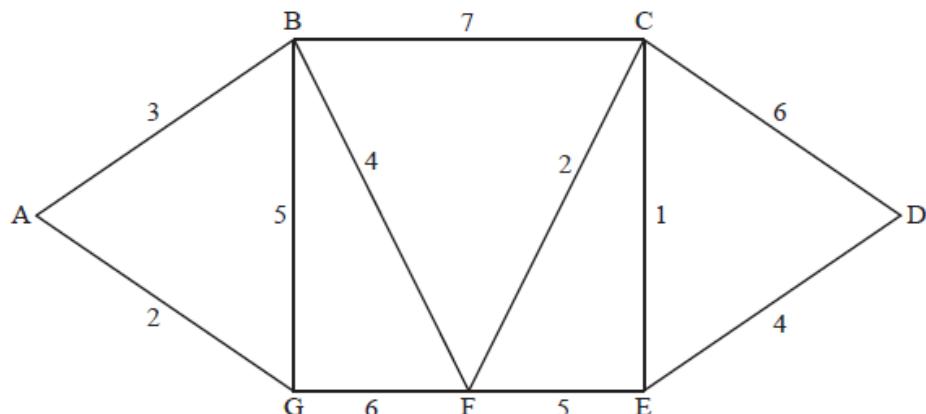
[4 marks]

Total [16 marks]

Examiners report

a. In part (a), a good number of candidates were able to 'see' the solution form for u_n and then (often in non-standard ways) successfully obtain

$u_n = 4 \times 7^n + 1$. A variety of methods and interesting approaches were seen here including use of the general closed form solution, iteration,

substitution of $u_n = 4 \times 7^n + 1$, substitution of $u_n = An + B$ and, interestingly, conversion to a second-degree linear recurrence relation. A

number of candidates erroneously converted the recurrence relation to a quadratic auxiliary equation and obtained $u_n = c_1(6)^n + c_2(1)^n$.

b. Compared to similar recurrence relation questions set in recent examination papers, part (b) was reasonably well attempted with a substantial

number of candidates correctly obtaining $v_n = 4(11)^n$. It was pleasing to note the number of candidates who could set up the correct auxiliary

equation and use the two given terms to obtain the required solution. It appeared that candidates were better prepared for solving second-order

linear recurrence relations compared to first-order linear recurrence relations.

c. Most candidates found part (c) challenging. Only a small number of candidates attempted to either factorise $11^n - 2^n$ or to subtract 7^n from the

expansion of $(7 + 4)^n$. It was also surprising how few went for the option of stating that 11 and 7 are congruent $\mod 4$ so $11^n - 7^n \equiv (\mod 4)$

and hence is a multiple of 4.

The diagram shows a weighted graph with vertices A, B, C, D, E, F, G. The weight of each edge is marked on the diagram.

a. (i)   Explain briefly why the graph contains an Eulerian trail but not an Eulerian circuit.   [4]

(ii)   Write down an Eulerian trail.

b. (i)   Use Dijkstra's algorithm to find the path of minimum total weight joining A to D.   [8]

(ii)   State the minimum total weight.

Markscheme

a. (i)   there is an Eulerian trail because there are only 2 vertices of odd degree   *R1*

there is no Eulerian circuit because not all vertices have even degree   *R1*

(ii)   *eg* GBAGFBCFECDE   *A2*

[4 marks]

b. (i)

Step	Vertices labelled	Working values	
1	A	A(0), B-3, G-2	*M1A1*
2	A, G	A(0), G(2), B-3, F-8	*A1*
3	A, G, B	A(0), G(2), B(3), F-7, C-10	*A1*
4	A, G, B, F	A(0), G(2), B(3), F(7), C-9, E-12	
5	A, G, B, F, C	A(0), G(2), B(3), F(7), C(9), E-10, D-15	*A1*
6	A, G, B, F, C, E	A(0), G(2), B(3), F(7), C(9), E(10), D-14	
7	A, G, B, F, C, E, D	A(0), G(2), B(3), F(7), C(9), E(10), D(14)	*A1*

Note: In both (i) and (ii) accept the tabular method including back tracking or labels by the vertices on a graph.

Note: Award *M1A1A1A1A0A0* if final labels are correct but intermediate ones are not shown.

(ii)   minimum weight path is ABFCED   *A1*

minimum weight is 14   *A1*

Note: Award the final two *A1* marks whether or not Dijkstra's Algorithm is used.

[8 marks]

Examiners report

a. In part (a) the criteria for Eulerian circuits and trails were generally well known and most candidates realised that they must start/finish at G/E. Candidates who could not do (a) generally struggled on the paper.

b. For part (b) the layout varied greatly from candidate to candidate. Not all candidates made their method clear and some did not show the temporary labels. It is recommended that teachers look at the tabular method with its backtracking system as it is an efficient way of tackling this type of problem and has a very clear layout.

(a)   Use the Euclidean algorithm to find gcd(12 306, 2976) .

(b)   Hence give the general solution to the diophantine equation $12\,306x + 2976y = 996$.

Markscheme

(a) $12\,306 = 4 \times 2976 + 402$ *M1*

$2976 = 7 \times 402 + 162$ *M1*

$402 = 2 \times 162 + 78$ *A1*

$162 = 2 \times 78 + 6$ *A1*

$78 = 13 \times 6$

therefore gcd is 6   **R1**

[5 marks]

(b) $6 | 996$ means there is a solution

$6 = 162 - 2(78)$ **(M1)(A1)**

$= 162 - 2\left(402 - 2(162)\right)$

$= 5(162) - 2(402)$ **(A1)**

$\boxed{= 5(2976) - 7(402)\} \text{\textbackslash right}) - 2(402)}$

$= 5(2976) - 37(402)$ **(A1)**

$= 5(2976) - 37\left(12\,306 - 4(2976)\right)$

$= 153(2976) - 37(12\,306)$ **(A1)**

$996 = 25\,398(2976) - 6142(12\,306)$

$\Rightarrow x_0 = -6142,\ y_0 = 25\,398$ **(A1)**

$\Rightarrow x = -6142 + \left(\frac{2976}{6}\right)t = -6142 + 496t$

$\Rightarrow y = 25\,398 - \left(\frac{12\,306}{6}\right)t = 25\,398 - 2051t$ **M1A1A1**

[9 marks]

Total [14 marks]

Examiners report

Part (a) of this question was the most accessible on the paper and was completed correctly by the majority of candidates. Most candidates were able to start part (b), but a number made errors on the way and quite a number failed to give the general solution.

---

a. Use the Euclidean algorithm to find the greatest common divisor of the numbers 56 and 315.   [4]

b. (i)   Find the general solution to the diophantine equation $56x + 315y = 21$.   [9]

(ii)   Hence or otherwise find the smallest positive solution to the congruence $315x \equiv 21$ (modulo 56) .

Markscheme

a. $315 = 5 \times 56 + 35$ **M1**

$56 = 1 \times 35 + 21$

$35 = 1 \times 21 + 14$ **A1**

$21 = 1 \times 14 + 7$

$14 = 2 \times 7$ **A1**

therefore gcd $= 7$ **A1**

[4 marks]

b. (i) $7 = 21 - 14$ *M1*

$= 21 - (35 - 21)$

$= 2 \times 21 - 35$ *(A1)*

$= 2 \times (56 - 35) - 35$

$= 2 \times 56 - 3 \times 35$ *(A1)*

$= 2 \times 56 - 3 \times (315 - 5 \times 56)$

$= 17 \times 56 - 3 \times 315$ *(A1)*

therefore $56 \times 51 + 315 \times (-9) = 21$ *M1*

$x = 51, \ y = -9$ is a solution *(A1)*

the general solution is $x = 51 + 45N$, $y = -9 - 8N$, $N \in \mathbb{Z}$ *A1A1*

(ii) putting $N = -2$ gives $y = 7$ which is the required value of x *A1*

[9 marks]

Examiners report

a. This question was generally well answered although some candidates were unable to proceed from a particular solution of the Diophantine equation to the general solution.

b. This question was generally well answered although some candidates were unable to proceed from a particular solution of the Diophantine equation to the general solution.

a. Given that a , $b \in \mathbb{N}$ and $c \in \mathbb{Z}^{+}$, show that if $a \equiv 1(\bmod\ c)$, then $ab \equiv b(\bmod\ c)$. [2]

b. Using mathematical induction, show that $9^{n} \equiv 1(\bmod\ 4)$, for $n \in \mathbb{N}$. [6]

c. The positive integer M is expressed in base 9. Show that M is divisible by 4 if the sum of its digits is divisible by 4. [4]

Markscheme

a. $a = \lambda c + 1$ *M1*

so $ab = \lambda bc + b \Rightarrow ab \equiv b(\bmod\ c)$ *A1AG*

[2 marks]

b. the result is true for $n = 0$ since $9^{0} = 1 \equiv 1(\bmod\ 4)$ *A1*

assume the result is true for $n = k$, *i.e.* $9^{k} \equiv 1(\bmod\ 4)$ *M1*

consider $9^{k+1} = 9 \times 9^{k}$ *M1*

$\equiv 9 \times 1(\bmod\ 4)$ **or** $1 \times 9^{k}(\bmod\ 4)$ *A1*

$\equiv 1(\bmod\ 4)$ *A1*

so true for $n = k \Rightarrow$ true for $n = k + 1$ and since true for $n = 0$ result follows by induction *R1*

Note: Do not award the final **R1** unless both **M1** marks have been awarded.

Note: Award the final **R1** if candidates state $n = 1$ rather than $n = 0$

[6 marks]

c. let $M = (a_n a_{n-1} \ldots a_0)_9$ **(M1)**

$= a_n \times 9^n + a_{n-1} \times 9^{n-1} + \ldots + a_0 \times 9^0$ **A1**

EITHER

$\equiv a_n (\bmod 4) + a_{n-1} (\bmod 4) + \ldots + a_0 (\bmod 4)$ **A1**

$\equiv \sum a_i (\bmod 4)$ **A1**

so M is divisible by 4 if $\sum a_i$ is divisible by 4   **AG**

OR

$= a_n(9^n - 1) + a_{n-1}(9^{n-1} - 1) + \ldots + a_1(9^1 - 1)$

$+ a_n + a_{n-1} + \ldots + a_1 + a_0$ **A1**

Since $9^n \equiv 1 (\bmod 4)$, it follows that $9^n - 1$ is divisible by 4,   **R1**

so M is divisible by 4 if $\sum a_i$ is divisible by 4   **AG**

[4 marks]

Examiners report

a. Part (a) was generally well answered. In (b), many candidates tested the result for $n = 1$ instead of $n = 0$. It has been suggested that the reason for this was a misunderstanding of the symbol N with some candidates believing it to denote the positive integers. It is important for candidates to be familiar with IB notation in which N denotes the positive integers and zero. In some scripts the presentation of the proof by induction was poor.

b. Part (a) was generally well answered. In (b), many candidates tested the result for $n = 1$ instead of $n = 0$. It has been suggested that the reason for this was a misunderstanding of the symbol N with some candidates believing it to denote the positive integers. It is important for candidates to be familiar with IB notation in which N denotes the positive integers and zero. In some scripts the presentation of the proof by induction was poor.

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Two mathematicians are planning their wedding celebration and are trying to arrange the seating plan for the guests. The only restriction is that all tables must seat the same number of guests and each table must have more than one guest. There are fewer than 350 guests, but they have forgotten the exact number. However they remember that when they try to seat them with two at each table there is one guest left over. The same

happens with tables of 3, 4, 5 and 6 guests. When there were 7 guests per table there were none left over. Find the number of guests.

Markscheme

let x be the number of guests

$x \equiv 1 (\bmod\ 2)$

$x \equiv 1 (\bmod\ 3)$

$x \equiv 1 (\bmod\ 4)$

$x \equiv 1 (\bmod\ 5)$

$x \equiv 1 (\bmod\ 6)$

$x \equiv 0 (\bmod\ 7)$ congruence (i)   *(M1)(A2)*

the equivalent of the first five lines is

$x \equiv 1\ (\bmod\ (\text{lcm of } 2, 3, 4, 5, 6)) \equiv 1 (\bmod\ 60)$ *A1*

$\Rightarrow x = 60t + 1$

from congruence (i) $60t + 1 \equiv 0 (\bmod\ 7)$ *M1A1*

$60t \equiv -1 (\bmod\ 7)$

$60t \equiv 6 (\bmod\ 7)$

$4t \equiv 6 (\bmod\ 7)$

$2t \equiv 3 (\bmod\ 7)$ *A1*

$\Rightarrow t = 7u + 5$ (or equivalent)   *A1*

hence $x = 420u + 300 + 1$ *A1*

$\Rightarrow x = 420u + 301$

smallest number of guests is 301   *A1*   *N6*

Note: Accept alternative correct solutions including exhaustion or formula from Chinese remainder theorem.

[10 marks]

Examiners report

There were a number of totally correct solutions to this question, but many students were unable to fully justify the result. Some candidates had learnt a formula to apply to the Chinese remainder theorem, but could not apply it well in this situation. Many worked with the conditions for divisibility but did not make much progress with the justification.

a. Show that a positive integer, written in base 10, is divisible by 9 if the sum of its digits is divisible by 9.   [7]

b. The representation of the positive integer N in base p is denoted by $(N)_p$.   [9]

If $(5^{(126)_7})_7 = (a_n a_{n-1} \ldots a_1 a_0)_7$, find a_0.

Markscheme

a. consider the decimal number $A = a_n a_{n-1} \ldots a_0$ *M1*

$A = A_n \times 10^n + a_{n-1} \times 10^{n-1} + \ldots + a_1 \times 10 + a_0$ *M1*

$= a_n \times (10^n - 1) + a_{n-1} \times (10^{n-1} - 1) + \ldots + a_1 \times (10 - 1) + a_n + a_{n-1} + \ldots + a_0$ *M1A1*

$= a_n \times 99 \ldots 9(n \text{ digits}) + a_{n-1} \times 99 \ldots 9(n - 1 \text{ digits}) + \ldots + 9a_1 + a_n + a_{n-1} + \ldots + a_0$ *A1*

all the numbers of the form $99 \ldots 9$ are divisible by 9 (to give $11 \ldots 1$),   *R1*

hence A is divisible by 9 if $\displaystyle\sum_{i=0}^{n} a_i$ is divisible by 9   *R1*

Note: A method that uses the fact that $10^t \equiv 1(\bmod\ 9)$ is equally valid.

[7 marks]

b. by Fermat's Little Theorem $5^6 \equiv 1(\bmod\ 7)$ *M1A1*

$(126)_7 = (49 + 14 + 6)_{10} = (69)_{10}$ *M1A1*

$5^{(126)_7} \equiv 5^{(11 \times 6 + 3)_{10}} \equiv 5^{(3)_{10}}(\bmod\ 7)$ *M1A1*

$5^{(3)_{10}} = (125)_{10} = (17 \times 7 + 6)_{10} \equiv 6(\bmod\ 7)$ *M1A1*

hence $a_0 = 6$ *A1*

[9 marks]

Examiners report

a. Questions similar to (a) have been asked in the past so it was surprising to see that solutions this time were generally disappointing.

b. In (b), most candidates changed the base 7 number 126 to the base 10 number 69. After that the expectation was that Fermat's little theorem would be used to complete the solution but few candidates actually did that. Many were unable to proceed any further and others used a variety of methods, for example working modulo 7,

$5^{69} = (5^2)^{34}.5 = 4^{34}.5 = (4^2)^{17}.5 = 2^{17}.5$ etc

This is of course a valid method, but somewhat laborious.

The positive integer N is expressed in base 9 as $(a_n a_{n-1} \ldots a_0)_9$.

(a)   Show that N is divisible by 3 if the least significant digit, a_0, is divisible by 3.

(b)   Show that N is divisible by 2 if the sum of its digits is even.

(c)   Without using a conversion to base 10, determine whether or not $(464860583)_9$ is divisible by 12.

Markscheme

(a)   let $N = a_n a_{n-1} \ldots a_1 a_0 = a_n \times 9^n + a_{n-1} \times 9^{n-1} + \ldots + a_1 \times 9 + a_0$ *M1A1*

all terms except the last are divisible by 3 and so therefore is their sum   *R1*

it follows that N is divisible by 3 if a_0 is divisible by 3   *AG*

[3 marks]

(b)   **EITHER**

consider N in the form

$$N = a_n \times (9^n - 1) + a_{n-1} \times (9^{n-1} - 1) + \ldots + a_1(9 - 1) + \sum_{i=1}^{n} a_i$$ *M1A1*

all terms except the last are even so therefore is their sum   *R1*

it follows that N is even if $\sum_{i=0}^{n} a_i$ is even   *AG*

OR

working modulo 2, $9^k \equiv 1 (\bmod\ 2)$ *M1A1*

hence $N = a_n a_{n-1} \ldots a_1 a_0 = a_n \times 9^n + a_{n-1} \times 9^{n-1} + \ldots + a_1 \times 9 + a_0 = \sum_{i=0}^{n} a_i (\bmod\ 2)$ *R1*

it follows that N is even if $\sum_{i=0}^{n} a_i$ is even   *AG*

[3 marks]

(c)   the number is divisible by 3 because the least significant digit is 3   *R1*

it is divisible by 2 because the sum of the digits is 44 which is even   *R1*

dividing the number by 2 gives $(232430286)_9$ *M1A1*

which is even because the sum of the digits is 30 which is even   *R1*

N is therefore divisible by a further 2 and is therefore divisible by 12   *R1*

Note: Accept alternative valid solutions.

[6 marks]

Total [12 marks]

Examiners report

Parts (a) and (b) were generally well answered. Part (c), however, caused problems for many candidates with some candidates even believing that

showing divisibility by 2 and 3 was sufficient to prove divisibility by 12. Some candidates stated that the fact that the sum of the digits was 44

(which itself is divisible by 4) indicated divisibility by 4 but this was only accepted if the candidates extended their proof in (b) to cover

divisibility by 4.

a. Explaining your method fully, determine whether or not 1189 is a prime number. [4]

b. (i) State the fundamental theorem of arithmetic. [6]

    (ii) The positive integers M and N have greatest common divisor G and least common multiple L. Show that $GL = MN$.

Markscheme

a. any clearly indicated method of dividing 1189 by successive numbers   *M1*

find that 1189 has factors 29 and/or 41   *A2*

it follows that 1189 is not a prime number   *A1*

Note: If no method is indicated, award *A1* for the factors and *A1* for the conclusion.

[4 marks]

b. (i) every positive integer, greater than 1, is either prime or can be expressed uniquely as a product of primes   *A1A1*

Note: Award *A1* for "product of primes" and *A1* for "uniquely".

(ii) **METHOD 1**

let M and N be expressed as a product of primes as follows

$M = AB$ and $N = AC$ *M1A1*

where A denotes the factors which are common and B, C the disjoint factors which are not common

it follows that $G = A$ *A1*

and $L = GBC$ *A1*

from these equations, it follows that

$GL = A \times ABC = MN$ *AG*

METHOD 2

Let $M = 2^{x_1} \times 3^{x_2} \times \ldots p_n^{x_n}$ and $N = 2^{y_1} \times 3^{y_2} \times \ldots p_n^{y_n}$ where p_n denotes the nth prime   *M1*

Then $G = 2^{\min(x_1,y_1)} \times 3^{\min(x_2,y_2)} \times \ldots p_n^{\min(x_n,y_n)}$ *A1*

and $L = 2^{\max(x_1,y_1)} \times 3^{\max(x_2,y_2)} \times \ldots p_n^{\max(x_n,y_n)}$ *A1*

It follows that $GL = 2^{x_1} \times 2^{y_1} \times 3^{x_2} \times 3^{y_2} \times \ldots \times p_n^{x_n} \times p_n^{y_n}$ *A1*

$= MN$ *AG*

[6 marks]

Examiners report

a. In (a), some candidates tried to use Fermat's little theorem to determine whether or not 1189 is prime but this method will not always work and in any case the amount of computation involved can be excessive. For this reason, it is strongly recommended that this method should not be used in examinations. In (b), it was clear from the scripts that candidates who had covered this material were generally successful and those who had not previously seen the result were usually unable to proceed.

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who had not previously seen the result were usually unable to proceed.

Andy and Roger are playing tennis with the rule that if one of them wins a game when serving then he carries on serving, and if he loses then the other player takes over the serve.

The probability Andy wins a game when serving is $\frac{1}{2}$ and the probability he wins a game when not serving is $\frac{1}{4}$. Andy serves in the first game. Let u_n denote the probability that Andy wins the nth game.

a. State the value of u_1. [1]

b. Show that u_n satisfies the recurrence relation [4]

$$u_n = \frac{1}{4}u_{n-1} + \frac{1}{4}.$$

c. Solve this recurrence relation to find the probability that Andy wins the nth game. [6]

d. After they have played many games, Pat comes to watch. Use your answer from part (c) to estimate the probability that Andy will win the game they are playing when Pat arrives. [2]

Markscheme

a. $\frac{1}{2}$ **A1**

   [1 mark]

b. Andy could win the nth game by winning the $n - 1^{\text{th}}$ and then winning the nth game or by losing the $n - 1^{\text{th}}$ and then winning the nth **(M1)**

   $u_n = \frac{1}{2}u_{n-1} + \frac{1}{4}(1 - u_{n-1})$ **A1A1M1**

   Note: Award **A1** for each term and **M1** for addition of two probabilities.

   $u_n = \frac{1}{4}u_{n-1} + \frac{1}{4}$ **AG**

   [4 marks]

c. general solution is $u_n = A\left(\frac{1}{4}\right)^n + p(n)$ **(M1)**

   for a particular solution try $p(n) = b$ **(M1)**
   $b = \frac{1}{4}b + \frac{1}{4}$ **(A1)**
   $b = \frac{1}{3}$
   hence $u_n = A\left(\frac{1}{4}\right)^n + \frac{1}{3}$ **(A1)**
   using $u_1 = \frac{1}{2}$ **M1**
   $\frac{1}{2} = A\left(\frac{1}{4}\right) + \frac{1}{3} \Rightarrow A = \frac{2}{3}$
   hence $u_n = \frac{2}{3}\left(\frac{1}{4}\right)^n + \frac{1}{3}$ **A1**

Note: Accept other valid methods.

[6 marks]

d. for large n $u_n \approx \frac{1}{3}$ *(M1)A1*

[2 marks]

Total [13 marks]

Examiners report

a. Not all candidates wrote this answer down correctly although it was essentially told you in the question.

b. Very badly answered. Candidates seemed to think that they were being told this relationship (so used it to find u(2)) rather than attempting to prove it.

c. This distinguished the better candidates. Some candidates though that they could use the method for homogeneous recurrence relations of second order and hence started solving a quadratic. Only the better candidates saw that it was a combined AP/GP.

d. The best candidates saw this but most had not done enough earlier to get to do this.

---

a. Use the pigeon-hole principle to prove that for any simple graph that has two or more vertices and in which every vertex is connected to at least   [4] one other vertex, there must be at least two vertices with the same degree.

b. Seventeen people attend a meeting.   [4]

Each person shakes hands with at least one other person and no-one shakes hands with

the same person more than once. Use the result from part (a) to show that there must be at least two people who shake hands with the same number of people.

c. Explain why each person cannot have shaken hands with exactly nine other people.   [2]

Markscheme

a. let there be v vertices in the graph; because the graph is simple the degree of each vertex is $\leq v - 1$ *A1*

the degree of each vertex is ≥ 1 *A1*

there are therefore $v - 1$ possible values for the degree of each vertex   *A1*

given there are v vertices by the pigeon-hole principle there must be at least two with the same degree   *R1*

[4 marks]

b. consider a graph in which the people at the meeting are represented by the vertices and two vertices are connected if the two people shake hands

   M1

the graph is simple as no-one shakes hands with the same person more than once (nor can someone shake hands with themselves)   *A1*

every vertex is connected to at least one other vertex as everyone shakes at least one hand   *A1*

the degree of each vertex is the number of handshakes so by the proof above there must be at least two who shake the same number of hands   *R1*

Note: Accept answers starting afresh rather than quoting part (a).

[4 marks]

c. (the handshaking lemma tells us that) the sum of the degrees of the vertices must be an even number   *A1*

the degree of each vertex would be 9 and 9×17 is an odd number (giving a contradiction)   *A1*

[2 marks]

Total [10 marks]

Examiners report

a. Generally there were too many "waffly" words and not enough precise statements leading to conclusions.

Misconceptions were: thinking that a few examples constituted a proof, thinking that the graph had to be connected, taking the edges as the pigeons not the degrees. The pigeon-hole principle was known but not always applied well.

b. Generally there were too many "waffly" words and not enough precise statements leading to conclusions.

Similar problems as in (a).

c. Generally there were too many "waffly" words and not enough precise statements leading to conclusions.

Many spurious reasons were given but good candidates went straight to the hand-shaking lemma.

---

a. A graph has n vertices with degrees 1, 2, 3, …, n. Prove that $n \equiv 0(\bmod\ 4)$ or $n \equiv 3(\bmod\ 4)$.   [6]

b. Let G be a simple graph with n vertices, $n \geqslant 2$. Prove, by contradiction, that at least two of the vertices of G must have the same degree.   [8]

Markscheme

a. as each edge contributes 1 to each of the vertices that it is incident with, each edge will contribute 2 to the sum of the degrees of all the vertices

   (R1)

so $2e = \sum \text{degrees}$ *(A1)*

$2e = \frac{n(n+1)}{2}$ *A1*

$4|n(n+1)$ *A1*

n and $n+1$ are coprime   *R1*

Note: Accept equivalent reasoning *e.g.* only one of n and $n + 1$ can be even.

$4|n$ or $4|n+1$ *A1*

$n \equiv 0(\bmod\ 4)$ or $n \equiv 3(\bmod\ 4)$ *AG*

[6 marks]

b. since G is simple, the highest degree that a vertex can have is $n-1$ *R1*

the degrees of the vertices must belong to the set $S = \{0,\ 1,\ 2,\ \ldots,\ n-1\}$ *A1*

proof by contradiction

if no two vertices have the same degree, all n vertices must have different degrees   *R1*

as there are only n different degrees in set S, the degrees must be precisely the n numbers 0, 1, 2, ..., $n - 1$ *R1*

let the vertex with degree 0 be A, then A is not adjacent to any of the other vertices   *R1*

let the vertex with degree $n - 1$ be B, then B is adjacent to all of the other vertices including A   *R1R1*

this is our desired contradiction, so there must be two vertices of the same degree   *R1*

[8 marks]

Examiners report

a. Only the top candidates were able to produce logically, well thought-out proofs.

b. Only the top candidates were able to produce logically, well thought-out proofs.

---

a. Use the Euclidean algorithm to find $\gcd(752, \ 352)$. [4]

b. A farmer spends £8128 buying cows of two different breeds, A and B, for her farm. A cow of breed A costs £752 and a cow of breed B [10] costs £352.

   (i)   Set up a diophantine equation to show this.

   (ii)   Using your working from part (a), find the general solution to this equation.

   (iii)   **Hence** find the number of cows of each breed bought by the farmer.

Markscheme

a. $752 = 2(352) + 48$ *M1*

   $352 = 7(48) + 16$ *A1*

   $48 = 3(16)$ *A1*

   therefore $\gcd(752, \ 352)$ is 16   *R1*

   [4 marks]

b. (i)   let x be the number of cows of breed A

   let y be the number of cows of breed B

   $752x + 352y = 8128$ *A1*

   (ii) $16 | 8128$ means there is a solution

   $16 = 352 - 7(48)$ *(M1)(A1)*

   $16 = 352 - 7\left(752 - 2(352)\right)$

   $16 = 15(352) - 7(752)$ *(A1)*

   $8128 = 7620(352) - 3556(752)$

   $\Rightarrow x_0 = -3556, \ y_0 = 7620$ *(A1)*

   $\Rightarrow x = -3556 + \left(\frac{352}{16}\right)t = -3556 + 22t$

$$\Rightarrow y = 7620 - \left(\frac{752}{16}\right) t = 7620 - 47t \quad \textbf{\textit{M1A1A1}}$$

(iii)   for x, y to be $\geqslant 0$, the only solution is $t = 162$ **M1**
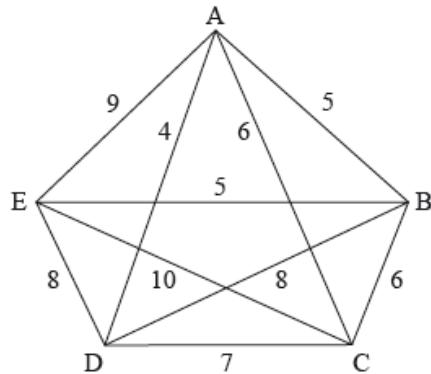
$\Rightarrow x = 8, \ y = 6$ **A1**

[10 marks]

Examiners report

a. Part (a) of this question was the most accessible on the paper and was completed correctly by the majority of candidates. It was pleasing to see that candidates were not put off by the question being set in context and most candidates were able to start part (b). However, a number made errors on the way, quite a number failed to give the general solution and it was only stronger candidates who were able to give a correct solution to part (b) (iii).

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The following diagram shows a weighted graph G with vertices A, B, C, D and E.

a. Show that graph G is Hamiltonian. Find the total number of Hamiltonian cycles in G, giving reasons for your answer.   [3]

b. State an upper bound for the travelling salesman problem for graph G.   [1]

d. Hence find a lower bound for the travelling salesman problem for G.   [2]

e. Show that the lower bound found in (d) cannot be the length of an optimal solution for the travelling salesman problem for the graph G.   [3]

Markscheme

a. *eg* the cycle A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ D $\rightarrow$ E $\rightarrow$ A is Hamiltonian   **A1**

starting from any vertex there are four choices
from the next vertex there are three choices, *etc* ...   **R1**

so the number of Hamiltonian cycles is 4! $(= 24)$ *A1*

Note: Allow 12 distinct cycles (direction not considered) or 60 (if different starting points count as distinct). In any case, just award the second *A1* if *R1* is awarded.
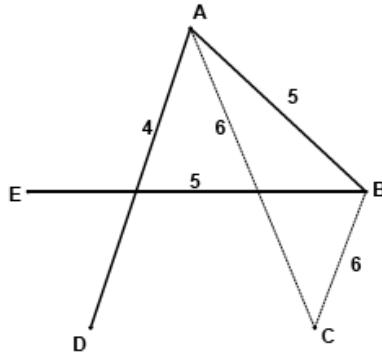
[3 marks]

b. total weight of any Hamiltonian cycles stated

eg $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A$ has weight $5 + 6 + 7 + 8 + 9 = 35$ *A1*
[1 mark]

d. a lower bound for the travelling salesman problem is then obtained by adding the weights of CA and CB to the weight of the minimum
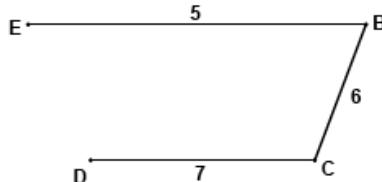
spanning tree   *(M1)*

a lower bound is then $14 + 6 + 6 = 26$ *A1*
[2 marks]
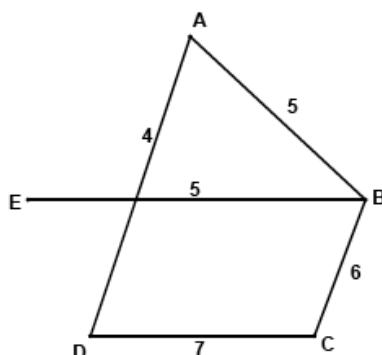
e. **METHOD 1**

eg eliminating A from *G,* a minimum spanning tree of weight 18 is   *(M1)*

   A1
adding AD and AB to the spanning tree gives a lower bound of $18 + 4 + 5 = 27 > 26$ *A1*

so 26 is not the best lower bound   *AG*

Note: Candidates may delete other vertices and the lower bounds obtained are B-28, D-27 and E-28.

METHOD 2
there are 12 distinct cycles (ignoring direction) with the following lengths

Cycle	Length	
ABCDEA	35	*M1*
ABCEDA	33	
ABDCEA	39	
ABDECA	37	
ABECDA	31	
ABEDCA	31	
ACBDEA	37	
ACBEDA	**29**	
ACDBEA	35	
ACEBDA	33	
AEBCDA	31	
AECBDA	37	*A1*

as the optimal solution has length 29   *A1*

26 is not the best possible lower bound   *AG*

Note: Allow answers where candidates list the 24 cycles obtained by allowing both directions.

[3 marks]

Examiners report

a. Part (a) was generally well answered, with a variety of interpretations accepted.

b. Part (b) also had a number of acceptable possibilities.

d. Part (d) was generally well answered, but there were few good attempts at part (e).

e. Part (d) was generally well answered, but there were few good attempts at part (e).

---

a. Define what is meant by the statement $x \equiv y(\bmod n)$ where $x, y, n \in \mathbb{Z}^{+}$.   [1]

b. Hence prove that if $x \equiv y(\bmod n)$ then $x^2 \equiv y^2(\bmod n)$.   [4]

c. Determine whether or not $x^2 \equiv y^2(\bmod n)$ implies that $x \equiv y(\bmod n)$.   [4]

Markscheme

a. $x \equiv y(\bmod n) \Rightarrow x = y + kn, \ (k \in \mathbb{Z})$ *A1*

[1 mark]

b. $x \equiv y(\bmod n)$

$\Rightarrow x = y + kn$ *M1*

$x^2 = y^2 + 2kny + k^2n^2$ *A1*

$\Rightarrow x^2 = y^2 + (2ky + k^2n)n$ *M1A1*

$\Rightarrow x^2 \equiv y^2(\bmod n)$ *AG*

[4 marks]

c. **EITHER**

$x^2 \equiv y^2(\bmod n)$

$\Rightarrow x^2 - y^2 = 0(\bmod\ n)$ **M1**

$\Rightarrow (x - y)(x + y) = 0(\bmod\ n)$ **A1**

This will be the case if

$x + y = 0(\bmod\ n)$ or $x = -y(\bmod\ n)$ **R1**

so $x \neq y(\bmod\ n)$ in general   **R1**

[4 marks]

OR

Any counter example, *e.g.* $n = 5,\ x = 3,\ y = 2$, in which case   **R2**

$x^2 \equiv y^2(\bmod\ n)$ but $x \neq y(\bmod\ n)$. (false)   **R1R1**

[4 marks]

Examiners report

a. While most candidates gave a correct meaning to $x \equiv y(\bmod\ n)$, there were some incorrect statements, the most common being

$x \equiv y(\bmod\ n)$ means that when x is divided by n, there is a remainder y. The true statement $8 \equiv 5(\bmod\ 3)$ shows that this statement is

incorrect. Part (b) was solved successfully by many candidates but (c) caused problems for some candidates who thought that the result in (c)

followed automatically from the result in (b).

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followed automatically from the result in (b).

---

a. Convert the decimal number 51966 to base 16.   [4]

b. (i)   Using the Euclidean algorithm, find the greatest common divisor, d , of 901 and 612.   [10]

(ii)   Find integers p and q such that $901p + 612q = d$.

(iii)   Find the least possible positive integers s and t such that $901s - 612t = 85$.

c. In each of the following cases find the solutions, if any, of the given linear congruence.   [5]

(i) $9x \equiv 3(\bmod\ 18)$

(ii) $9x \equiv 3(\bmod\ 15)$

Markscheme

a. the relevant powers of 16 are 16, 256 and 4096

then

$51966 = 12 \times 4096$ remainder 2814   **M1A1**

$2814 = 10 \times 256$ remainder 254

$254 = 15 \times 16$ remainder 14   **A1**

the hexadecimal number is CAFE   **A1**

Note: CAFE is produced using a standard notation, accept explained alternative notations.

[4 marks]

b. (i)   using the Euclidean Algorithm   **(M1)**

$901 = 612 + 289$ **(A1)**

$612 = 2 \times 289 + 34$

$289 = 8 \times 34 + 17$

$\gcd(901,\ 612) = 17$ **A1**

(ii)   working backwards   **(M1)**

$17 = 289 - 8 \times 34$

$= 289 - 8 \times (612 - 2 \times 289)$

$= 17 \times (901 - 612) - 8 \times 612$

$= 27 \times 901 - 25 \times 612$

so $p = 17,\ q = -25$ **A1A1**

(iii)   a particular solution is

$s = 5p = 85,\ t = -5q = 125$ **(A1)**

the general solution is

$s = 85 + 36\lambda,\ t = 125 + 53\lambda$ **M1A1**

by inspection the solution satisfying all conditions is

$(\lambda = -2),\ s = 13,\ t = 19$ **A1**

[10 marks]

c. (i)   the congruence is equivalent to $9x = 3 + 18\lambda$ **(A1)**

this has no solutions as 9 does not divide the RHS   **R1**

(ii)   the congruence is equivalent to $3x = 1 + 5\lambda,\ (3x \equiv 1(\bmod 5))$ **A1**

one solution is $x = 2$, so the general solution is $x = 2 + 5n$ $(x \equiv 2(\bmod 5))$ **M1A1**

[5 marks]

Examiners report

a. Many did not seem familiar with hexadecimal notation and often left their answer as 12101514 instead of CAFE.

b. The Euclidean algorithm was generally found to be easy to deal with but getting a general solution in part (iii) eluded many candidates.

c. Rewriting the congruence in the form $9x = 3 + 18\lambda$ for example was not often seen but should have been the first thing thought of.

Given that $a,\ b,\ c,\ d \in \mathbb{Z}$, show that

$$(a - b)(a - c)(a - d)(b - c)(b - d)(c - d) \equiv 0(\bmod 3).$$

Markscheme

Examiners report

Most candidates who solved this question used the argument that there are four variables which can take only one of three different values modulo 3 so that at least two must be equivalent modulo 3 which leads to the required result. This apparently simple result, however, requires a fair amount of insight and few candidates managed it.

---

a. The sequence $\{u_n\}$, $n \in \mathbb{Z}^+$, satisfies the recurrence relation $u_{n+2} = 5u_{n+1} - 6u_n$.   [6]

Given that $u_1 = u_2 = 3$, obtain an expression for u_n in terms of n .

b. The sequence $\{v_n\}$, $n \in \mathbb{Z}^+$, satisfies the recurrence relation $v_{n+2} = 4v_{n+1} - 4v_n$.   [8]

Given that $v_1 = 2$ and $v_2 = 12$, use the principle of strong mathematical induction to show that $v_n = 2^n(2n - 1)$.

Markscheme

a. the auxiliary equation is

$m^2 - 5m + 6 = 0$ ***M1***

giving $m = 2, 3$ ***A1***

the general solution is

$u_n = A \times 2^n + B \times 3^n$ ***A1***

substituting $n = 1, 2$

$2A + 3B = 3$ ***M1***

$4A + 9B = 3$ ***A1***

the solution is $A = 3$, B $= -1$ giving $u_n = 3 \times 2^n - 3^n$ ***A1***

[6 marks]

b. we first prove that $v_n = 2^n(2n - 1)$ for $n = 1, 2$ ***M1***

for $n = 1$, it gives $2 \times 1 = 2$ which is correct

for $n = 2$, it gives $4 \times 3 = 12$ which is correct   ***A1***

we now assume that the result is true for $n \leqslant k$ ***M1***

consider

$v_{k+1} = 4v_k - 4v_{k-1} \ (k \geqslant 2)$ ***M1***

$= 4.2^k(2k - 1) - 4.2^{k-1}(2k - 3)$ ***A1***

$= 2^{k+1}(4k - 2 - 2k + 3)$ ***A1***

$= 2^{k+1}(2(k+1) - 1)$ ***A1***

this proves that if the result is true for $n \leqslant k$ then it is true for $n \leqslant k + 1$

since we have also proved it true for $n \leqslant 2$, the general result is proved by induction   ***R1***

Note: A reasonable attempt has to be made to the induction step for the final ***R1*** to be awarded.

[8 marks]

Examiners report

---

a. State the Fundamental theorem of arithmetic for positive whole numbers greater than 1.   [2]

b. Use the Fundamental theorem of arithmetic, applied to 5577 and $99\,099$, to calculate $\gcd(5577, \ 99\,099)$ and $\operatorname{lcm}(5577, \ 99\,099)$, expressing   [3]

each of your answers as a product of prime numbers.

c. Prove that $\gcd(n, \ m) \times \operatorname{lcm}(n, \ m) = n \times m$ for all $n, \ m \in \mathbb{Z}^+$.   [6]

Markscheme

a. every positive integer, greater than 1, is either prime or can be expressed uniquely as a product of primes   ***A1A1***

Note: Award ***A1*** for "product of primes" and ***A1*** for "uniquely".

[2 marks]

b. $5577 = 3 \times 11 \times 13^2$ and $99099 = 3^2 \times 7 \times 11^2 \times 13$ **M1**

  $\gcd(5577,\ 99099) = 3 \times 11 \times 13$, $\operatorname{lcm}(5577,\ 99099) = 3^2 \times 7 \times 11^2 \times 13^2$ **A1A1**

  [3 marks]

c. **METHOD 1**

  $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ and $m = p_1^{j_1} p_2^{j_2} \ldots p_r^{j_r}$ **M1**

  employing all the prime factors of n and m, and inserting zero exponents if necessary   **R1**

  define $g_i = \min(k_i,\ j_i)$ and $h_i = \max(k_i,\ j_i)$ for $i = 1 \ldots r$ **(M1)**

  $\gcd(n,\ m) = p_1^{g_1} p_2^{g_2} \ldots p_r^{g_r}$ and $\operatorname{lcm}(n,\ m) = p_1^{h_1} p_2^{h_2} \ldots p_r^{h_r}$ **A1A1**

  noting that $g_i + h_i = k_i + j_i$ for $i = 1 \ldots r$ **R1**

  $\gcd(n,\ m) \times \operatorname{lcm}(n,\ m) = n \times m$ for all $n,\ m \in \mathbb{Z}^+$ **AG**

  METHOD 2

  let m and n be expressed as a product of primes where $m = ab$ and $n = ac$ **M1**

  a denotes the factors that are common and $b,\ c$ are the disjoint factors that are not common   **R1**

  $\gcd(n,\ m) = a$ **A1**

  $\operatorname{lcm}(n,\ m) = \gcd(n,\ m)bc$ **A1**

  $\gcd(n,\ m) \times \operatorname{lcm}(n,\ m) = a \times (abc)$ **M1**

  $= ab \times ac$ and $m = ab$ and $n = ac$ so   **R1**

 $\gcd(n,\ m) \times 1\operatorname{cm}(n,\ m) = n \times m$ for all $n,\ m \in \mathbb{Z}^+$ **AG**
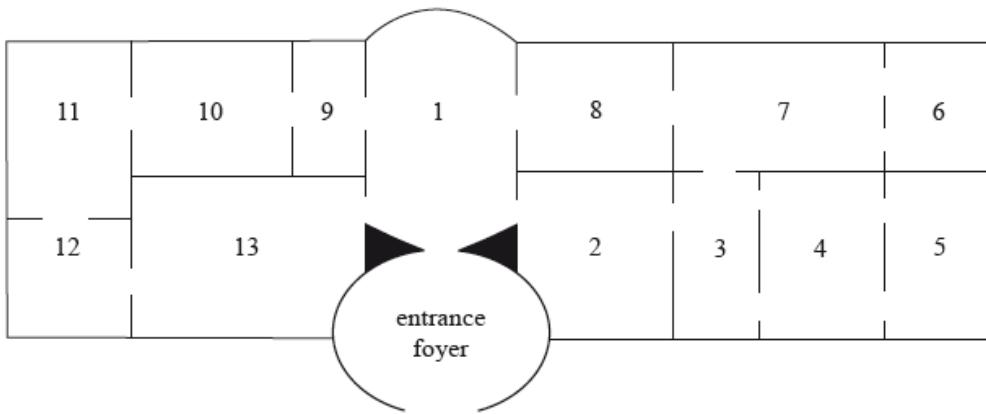
  [6 marks]

  Total [11 marks]

Examiners report

a. In part (a), most candidates omitted the 'uniquely' in their definition of the fundamental theorem of arithmetic. A few candidates defined what a prime number is.

b. In part (b), a substantial number of candidates used the Euclidean algorithm rather than the fundamental theorem of arithmetic to calculate $\gcd(5577,\ 99099)$ and $\operatorname{lcm}(5577,\ 99099)$.

c. In part (c), a standard proof that has appeared in previous examination papers, was answered successfully by candidates who were well prepared.

The following figure shows the floor plan of a museum.

(a) (i) Draw a graph G that represents the plan of the museum where each exhibition room is represented by a vertex labelled with the exhibition room number and each door between exhibition rooms is represented by an edge. Do not consider the entrance foyer as a museum exhibition room.

(ii) Write down the degrees of the vertices that represent each exhibition room.

(iii) Virginia enters the museum through the entrance foyer. Use your answers to (i) and (ii) to justify why it is possible for her to visit the thirteen exhibition rooms going through each internal doorway exactly once.

(b) Let G and H be two graphs whose adjacency matrices are represented below.
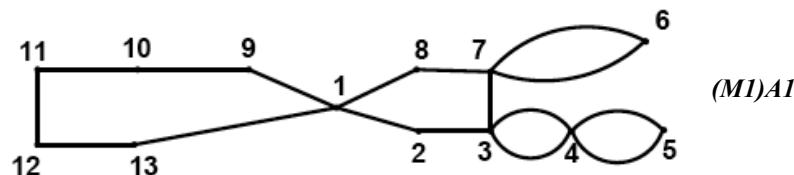
G

	A	B	C	D	E	F
A	0	2	0	2	0	0
B	2	0	1	1	0	1
C	0	1	0	1	2	1
D	2	1	1	0	2	0
E	0	0	2	2	0	2
F	0	1	1	0	2	0

H

	P	Q	R	S	T	U
P	0	1	3	0	1	2
Q	1	0	1	3	2	0
R	3	1	0	2	1	3
S	0	3	2	0	2	0
T	1	2	1	2	0	1
U	2	0	3	0	1	0

Using the adjacency matrices,

(i) find the number of edges of each graph;

(ii) show that exactly one of the graphs has a Eulerian trail;

(iii) show that neither of the graphs has a Eulerian circuit.

Markscheme

(a) (i)

(M1)A1

Note: Do not penalize candidates who include the entrance foyer.

(ii) the degrees of the vertices are 4, 2, 4, 4, 2, 2, 4, 2, 2, 2, 2, 2, 2 *A1*

(iii) the degree of all vertices is even and hence a Eulerian circuit exists, *A1*

hence it is possible to enter the museum through the foyer and visit each room 1–13 going through each internal doorway exactly once *AG*

Note: The connected graph condition is not required.

(b) (i)

G

	A	B	C	D	E	F	total
A	0	2	0	2	0	0	4
B	2	0	1	1	0	1	5
C	0	1	0	1	2	1	5
D	2	1	1	0	2	0	6
E	0	0	2	2	0	2	6
F	0	1	1	0	2	0	4
total							30

H

	P	Q	R	S	T	U	total
P	0	1	3	0	1	2	7
Q	1	0	1	3	2	0	7
R	3	1	0	2	1	3	10
S	0	3	2	0	2	0	7
T	1	2	1	2	0	1	7
U	2	0	3	0	1	0	6
total							44

(M1)

graph G has 15 edges and graph H has 22 edges   *A1A1*

(ii)   the degree of every vertex is equal to the sum of the numbers in the corresponding row (or column) of the adjacency table

exactly two of the vertices of G have an odd degree (B and C)   *A1*

H has four vertices with odd degree   *A1*

G is the graph that has a Eulerian trail (and H does not)   *R1*

(iii)   neither graph has all vertices of even degree   *R1*

therefore neither of them has a Eulerian circuit   *AG*

[7 marks]

Examiners report

Part (a) was generally well answered. There were many examples of full marks in this part. Part (b) caused a few more difficulties, although there were many good solutions. Few candidates used the matrix to find the number of edges, preferring instead to draw the graph. A surprising number of students confused the ideas of having vertices of odd degree.

---

a. Show that 30 is a factor of $n^5 - n$ for all $n \in \mathbb{N}$. [5]

b. (i)   Show that $3^{3^m} \equiv 3 \pmod 4$ for all $m \in \mathbb{N}$. [8]

(ii)   Hence show that there is exactly one pair $(m,\ n)$ where $m,\ n \in \mathbb{N}$, satisfying the equation $3^{3^m} = 2^{2^n} + 5^2$.

Markscheme

a. **METHOD 1**

$n^5 - n = \underbrace{n(n-1)(n+1)}_{\text{3 consecutive integers}}(n^2+1) \equiv 0 \pmod 6$ *M1*

at least a factor is multiple of 3 and at least a factor is multiple of 2   *R1*

$n^5 - n = n(n^4 - 1) \equiv 0 \pmod 5$ as $n^4 \equiv 1 \pmod 5$ by FLT   *R1*

therefore, as $(5,\ 6) = 1$,   *R1*

$n^5 - n \equiv 0 \left(\mod \underbrace{5 \times 6}_{30} \right)$ *A1*

ie 30 is a factor of $n^5 - n$ *AG*

METHOD 2

let $\text{P}(n)$ be the proposition: $n^5 - n = 30\alpha$ for some $\alpha \in \mathbb{Z}$

$0^5 - 0 = 30 \times 0$, so $\text{P}(0)$ is true   *A1*

assume $P(k)$ is true for some k and consider $P(k+1)$

$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$ *M1*

$= (k^5 - k) + 5k \left(\underbrace{k^3 + 3k^2 + 3k + 1}_{(k+1)^3} - (k^2 + k) \right)$

$= (k^5 - k) + 5k \left((k+1)^3 - k(k+1) \right)$

$= 30\alpha + \underbrace{5k(k+1) \left(\underbrace{k^2 + k + 1}_{k(k+1)+1} \right)}_{\text{multiple of 6}}$ *M1*

$= 30\alpha + 30\beta$ *A1*

as $P(0)$ is true and $P(k)$ true implies $P(k+1)$ true, by PMI $P(n)$ is true for all values $n \in \mathbb{N}$ *R1*

Note: Award the first *M1* only if the correct induction procedure is followed and the correct first line is seen.

Note: Award *R1* only if both *M* marks have been awarded.

METHOD 3

$n^5 - n = n(n^4 - 1)$ *M1*

$= n(n^2 - 1)(n^2 + 1)$ *A1*

$= (n - 1)n(n + 1)(n^2 - 4 + 5)$ *R1*

$= (n - 2)(n - 1)n(n + 1)(n + 2) + 5(n - 1)n(n + 1)$ *A1*

each term is multiple of 2, 3 and 5   *R1*

therefore is divisible by 30   *AG*

[5 marks]

b. (i)   **METHOD 1**

case 1: $m = 0$ and $3^{3^0} \equiv 3\bmod 4$ is true   *A1*

case 2: $m > 0$

let $N = 3^m \geqslant 3$ and consider the binomial expansion   *M1*

$3^N = (1 + 2)^N = \sum_{k=0}^{N} \binom{N}{k} 2^k = 1 + 2N + \underbrace{\sum_{k=2}^{N} \binom{N}{k} 2^k}_{\equiv 0 (\bmod 4)} \equiv 1 + 2N (\bmod 4)$ *A1*

as $\underbrace{3^m}_{N} \equiv (-1)^m (\bmod 4) \Rightarrow 1 + 2N \equiv 1 + 2(-1)^m (\bmod 4)$ *R1*

therefore $\underbrace{3^{3^m}}_{3^N} \equiv 1 + 2(-1)^m (\bmod 4) \Rightarrow \begin{cases} \underbrace{3^{3^m}}_{3^N} \equiv \underbrace{1 + 2}_{3}(\bmod 4) \text{ for } m \text{ even} \\ \underbrace{3^{3^m}}_{3^N} \equiv \underbrace{1 - 2}_{-1 \equiv 3(\bmod 4)} (\bmod 4) \text{ for } m \text{ odd} \end{cases}$ *R1*

which proves that $3^{3^m} \equiv 3(\bmod 4)$ for any $m \in \mathbb{N}$ *AG*

METHOD 2

let $P(n)$ be the proposition: $3^{3^n} - 3 \equiv 0(\bmod 4, \text{ or } 24)$

$3^{3^0} - 3 = 3 - 3 \equiv 0(\bmod 4 \text{ or } 24)$, so $P(0)$ is true   *A1*

assume $P(k)$ is true for some k *M1*

consider $3^{3^{k+1}} - 3 = 3^{3^k \times 3} - 3$ *M1*

$= (3 + 24r)^3 - 3$

$\equiv 27 + 24t - 3$ *R1*

$\equiv 0(\bmod 4 \text{ or } 24)$

as $P(0)$ is true and $P(k)$ true implies $P(k+1)$ true, by PMI $P(n)$ is true for all values $n \in \mathbb{N}$ *R1*

METHOD 3

$3^{3^m} - 3 = 3(3^{3^m - 1} - 1)$ *M1A1*

$= 3(3^{2k} - 1)$ *R1*

$= 3(9^k - 1)$

$$= 3\underbrace{\left((8+1)^k - 1\right)}_{\text{multiple of 8}} \quad \textbf{\textit{R1}}$$

$\equiv 0 \pmod{24} \quad \textbf{\textit{A1}}$

which proves that $3^{3^m} \equiv 3 \pmod 4$ for any $m \in \mathbb{N}$ **AG**

(ii)   for $m \in \mathbb{N}$, $3^{3^m} \equiv 3 \pmod 4$ and, as $2^{2^n} \equiv 0 \pmod 4$ and $5^2 \equiv 1 \pmod 4$ then $2^{2^n} + 5^2 \equiv 1 \pmod 4$ for $n \in \mathbb{Z}^+$

there is no solution to $3^{3^m} = 2^{2^n} + 5^2$ for pairs $(m,\ n) \in \mathbb{N} \times \mathbb{Z}^+$ **R1**

when $n = 0$, we have $3^{3^m} = 2^{2^0} + 5^2 \Rightarrow 3^{3^m} = 27 \Rightarrow m = 1$ **M1**

therefore $(m,\ n) = (1,\ 0)$ **A1**

is the only pair of non-negative integers that satisfies the equation   **AG**

[8 marks]
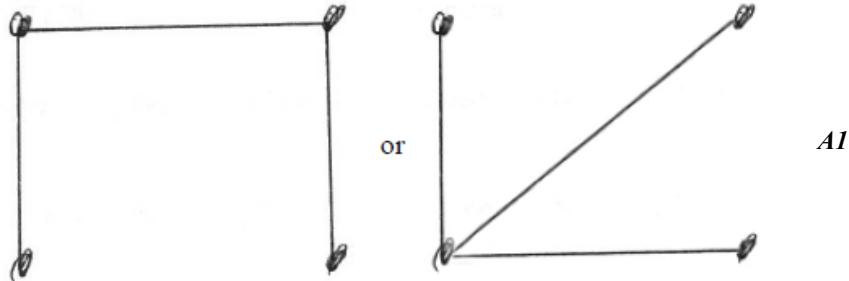
Examiners report

a. Many students were able to get partial credit for part (a), although few managed to gain full marks.

b. There seemed to be very few good attempts at part (b), many failing at the outset to understand what was meant by 3^{3^m}.

(a)   Draw a spanning tree for

      (i)   the complete graph, K_4;
      (ii)   the complete bipartite graph, $K_{4,4}$.

(b)   Prove that a simple connected graph with n vertices, where $n > 1$, must have two vertices of the same degree.

(c)   Prove that every simple connected graph has at least one spanning tree.

Markscheme

(a)   (i)

      or   **A1**
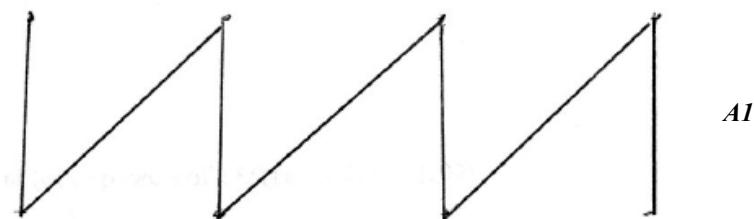
Note: Or equivalent not worrying about the orientation.

(ii)

      A1

Note: Other trees are possible, but must clearly come from the bipartite graph, so, for example, a straight line graph is not acceptable unless the bipartite nature is clearly shown *eg*, with black and white vertices.

[2 marks]

(b)   graph is simple implies maximum degree is $n - 1$ *A1*
graph is connected implies minimum degree is 1   *A1*
by a pigeon-hole principle two vertices must have the same degree   *R1*
[3 marks]

(c)   if the graph is not a tree it contains a cycle   *A1*
remove one edge of this cycle   *M1*
the graph remains connected   *A1*
repeat until there are no cycles   *M1*
the final graph is connected and has no cycles   *A1*
so is a tree   *AG*

Note: Allow other methods *eg*, induction, reference to Kruskal's algorithm.

[5 marks]

Total [10 marks]

Examiners report

The weights of the edges in the complete graph G are shown in the following table.

	A	B	C	D	E	F
A	–	14	10	8	12	9
B	14	–	9	12	10	13
C	10	9	–	7	8	13
D	8	12	7	–	9	11
E	12	10	8	9	–	11
F	9	13	13	11	11	–

a. Starting at A, use the nearest neighbour algorithm to find an upper bound for the travelling salesman problem for G.   [5]

b. By first removing A , use the deleted vertex algorithm to find a lower bound for the travelling salesman problem for G.   [7]

Markscheme

a. attempt to use the nearest neighbour algorithm   *M1*

the nearest neighbour path is $A \rightarrow D \rightarrow C$ $A1$
$\rightarrow E \rightarrow B \rightarrow F \rightarrow A$ $A1$

the upper bound is the total weight of this path, *ie*   *(M1)*

$8 + 7 + 8 + 10 + 13 + 9 = 55$ **A1**

Note: The **(M1)** is for adding 6 weights together.

[5 marks]

b. attempt to use an appropriate algorithm, with A deleted, to determine the minimum spanning tree, *eg* Kruskal's **(M1)**

CD (7) **A1**

CE, CB (8,9) **A1**

DF or EF (11) **A1**

the weight of this minimum spanning tree is 35 **(A1)**

adding in the two smallest weights joining A (AD and AF) to this tree gives a lower bound **(M1)**

of $35 + 8 + 9 = 52$ **A1**
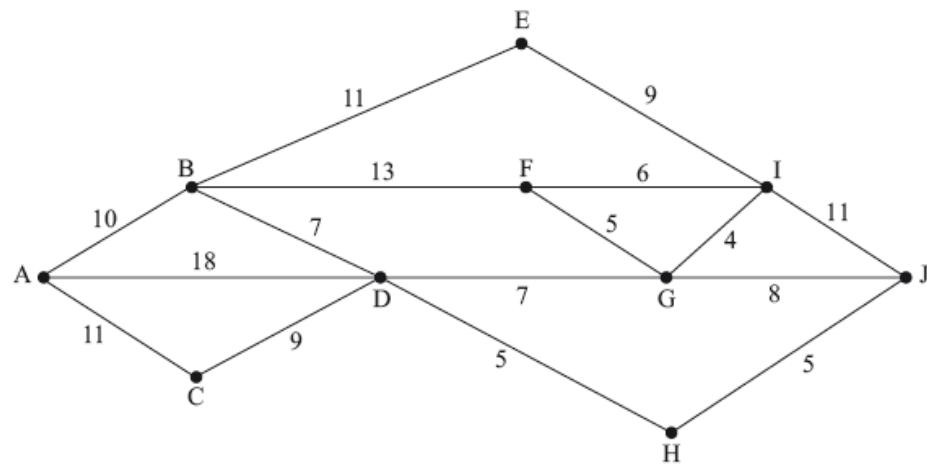
Note: Clear diagrams aiding solutions are acceptable in (a) and (b).

[7 marks]

Examiners report

a. Generally good use of the nearest neighbour algorithm. Some candidates showed no knowledge of it and there was some confusion with the twice

the weight of a minimal spanning tree method. Some candidates forgot to go back to A and thus did not have a Hamiltonian cycle.

b. The method was generally known. Some candidates used nearest neighbour instead of Kruskal's algorithm to find a minimal spanning tree. Some

forgot to add in the two edges connected to . Some with the right method made mistakes in not noticing the correct edge to choose. A

The following graph represents the cost in dollars of travelling by bus between 10 towns in a particular province.

a. Use Dijkstra's algorithm to find the cheapest route between A and J, and state its cost.   [7]

b. For the remainder of the question you may find the cheapest route between any two towns by inspection.   [6]

It is given that the total cost of travelling on all the roads without repeating any is 139.

A tourist decides to go over all the roads at least once, starting and finishing at town A.

Find the lowest possible cost of his journey, stating clearly which roads need to be travelled more than once. You must fully justify your answer.

Markscheme

a.

	A	B	C	D	E	F	G	H	I	J
A	0	←10	11	18						
B		10	←—17		21	23				
C			11							
D				17	←—		24	22		
E					21				30	
H								22	←—	27
F						23			29	
G							24		28	
J										27

M1A1A1A1

(**M1** for an attempt at Dijkstra's)

(**A1** for value of $D = 17$)

(**A1** for value of $H = 22$)

(**A1** for value of $G = 22$)

route is $ABDHJ$ **(M1)A1**

cost is $27   **A1**

Note: Accept other layouts.

[7 marks]

b.   there are 4 odd vertices A, D, F and J **A1**

these can be joined up in 3 ways with the following extra costs

AD and FJ $17 + 13 = 30$

AF and DJ $23 + 10 = 33$

AJ and DF $27 + 12 = 39$ **M1A1A1**

Note: Award **M1** for an attempt to find different routes.

Award **A1A1** for correct values for all three costs **A1** for one correct.

need to repeat AB, BD, FG and GJ **A1**

total cost is $139 + 30 = \$169$ **A1**

[6 marks]

Total [13 marks]

Examiners report

a.   Many candidates had the correct route and the cost. Not all showed sufficient working with their Dijkstra's algorithm. See the mark-scheme for the

neat way of laying out the working, including the back-tracking method. This tabular working is efficient, avoids mistakes and saves time.

b. There was often confusion here between this problem and the travelling salesman. Good candidates started with the number of vertices of odd degree. Weaker candidates just tried to write the answer down without complete reasoning. All the 3 ways of joining the odd vertices had to be considered so that you knew you had the smallest. Sometimes a mark was lost by giving which routes (paths) had to be repeated rather than which roads (edges).

---

a. Use the Euclidean algorithm to find the greatest common divisor of 259 and 581. [4]

b. Hence, or otherwise, find the general solution to the diophantine equation $259x + 581y = 7$. [5]

Markscheme

a. $581 = 2 \times 259 + 63$ *M1A1*

$259 = 4 \times 63 + 7$ *A1*

$63 = 9 \times 7$

the GCD is therefore 7   *A1*

[4 marks]

b. consider

$7 = 259 - 4 \times 63$ *M1*

$= 259 - 4 \times (581 - 2 \times 259)$ *A1*

$= 259 \times 9 + 581 \times (-4)$ *A1*

the general solution is therefore

$x = 9 + 83n; \ y = -4 - 37n$ where $n \in \mathbb{Z}$ *M1A1*

Notes: Accept solutions laid out in tabular form. Dividing the diophantine equation by 7 is an equally valid method.

[5 marks]

Examiners report

a. [N/A]

b. [N/A]

An arithmetic sequence has first term 2 and common difference 4. Another arithmetic sequence has first term 7 and common difference 5. Find the set of all numbers which are members of both sequences.

Markscheme

the mth term of the first sequence $= 2 + 4(m - 1)$ *(M1)(A1)*

the nth term of the second sequence $= 7 + 5(n-1)$ *(A1)*

EITHER

equating these, *M1*

$5n = 4m - 4$

$5n = 4(m-1)$ *(A1)*

4 and 5 are coprime *(M1)*

$\Rightarrow 4|n$ so $n = 4s$ or $5|(m-1)$ so $m = 5s + 1$, $s \in \mathbb{Z}^+$ *(A1)A1*

thus the common terms are of the form $\{2 + 20s; \ s \in \mathbb{Z}^+\}$ *A1*

OR

the numbers of both sequences are

2, 6, 10, 14, 18, 22

7, 12, 17, 22 *A1*

so 22 is common *A1*

identify the next common number as 42 *(M1)A1*

the general solution is $\{2 + 20s; \ s \in \mathbb{Z}^+\}$ *(M1)A1*

[9 marks]

Examiners report

Solutions to this question were extremely variable with some candidates taking several pages to give a correct solution and others taking several pages and getting nowhere. Some elegant solutions were seen including the fact that the members of the two sets can be represented as 2 mod 4 and 2 mod 5 respectively so that common members are 2 mod 20.

---

a. Using the Euclidean algorithm, show that $\gcd(99, \ 332) = 1$.   [4]

b. (i)   Find the general solution to the diophantine equation $332x - 99y = 1$.   [11]

(ii)   Hence, or otherwise, find the smallest positive integer satisfying the congruence $17z \equiv 1(\bmod 57)$.

Markscheme

a. using the Euclidean Algorithm,

$332 = 3 \times 99 + 35$ *M1*

$99 = 2 \times 35 + 29$ *A1*

$35 = 1 \times 29 + 6$

$29 = 4 \times 6 + 5$ *A1*

$6 = 1 \times 5 + 1$ *A1*

hence 332 and 99 have a gcd of 1   *AG*

Note: For both (a) and (b) accept layout in tabular form, especially the brackets method of keeping track of the linear combinations as the method proceeds.

[4 marks]

b. (i)   working backwards,   *(M1)*

$6 - 5 = 1$

$6 - (29 - 4 \times 6) = 1$ or $5 \times 6 - 29 = 1$ *A1*

$5 \times (35 - 29) - 29 = 1$ or $5 \times 35 - 6 \times 29 = 1$ *A1*

$5 \times 35 - 6 \times (99 - 2 \times 35) = 1$ or $17 \times 35 - 6 \times 99 = 1$

$17 \times (332 - 3 \times 99) - 6 \times 99 = 1$ or $17 \times 332 - 57 \times 99 = 1$ *A1*

a solution to the Diophantine equation is therefore

$x = 17, \ y = 57$ *(A1)*

the general solution is

$x = 17 + 99N, \ y = 57 + 332N$ *A1A1*

Note: If part (a) is wrong it is inappropriate to give *FT* in (b) as the numbers will contradict, however the *M1* can be given.

(ii)   it follows from previous work that

$17 \times 332 = 1 + 99 \times 57$ *(M1)*

$\equiv 1(\bmod 57)$ *(A1)*

$z = 332$ is a solution to the given congruence   *(A1)*

the general solution is $332 + 57N$ so the smallest solution is 47   *A1*

[11 marks]

Examiners report

a. Part (a) was well answered and (b) fairly well answered. There were problems with negative signs due to the fact that there was a negative in the question, so candidates had to think a little, rather than just remembering formulae by rote. The lay-out of the algorithm that keeps track of the linear combinations of the first 2 variables is recommended to teachers.

b. Part (a) was well answered and (b) fairly well answered. There were problems with negative signs due to the fact that there was a negative in the question, so candidates had to think a little, rather than just remembering formulae by rote. The lay-out of the algorithm that keeps track of the linear combinations of the first 2 variables is recommended to teachers.

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a. Solve, by any method, the following system of linear congruences   [3]

$x \equiv 9(\bmod 11)$

$$x \equiv 1(\bmod 5)$$

b. Find the remainder when 41^{82} is divided by 11. [4]

c. Using your answers to parts (a) and (b) find the remainder when 41^{82} is divided by 55. [3]

Markscheme

a. **METHOD 1**

listing $9, 20, 31, \ldots$ and $1, 6, 11, 16, 21, 26, 31, \ldots$ ___M1___

one solution is 31 ___(A1)___

by the Chinese remainder theorem the full solution is

$x \equiv 31(\bmod 55)$ ___A1 N2___

METHOD 2

$x \equiv 9(\bmod 11) \Rightarrow x = 9 + 11t$ ___M1___

$\Rightarrow 9 + 11t \equiv 1(\bmod 5)$

$\Rightarrow t \equiv 2(\bmod 5)$ ___A1___

$\Rightarrow t = 2 + 5s$

$\Rightarrow x = 9 + 11(2 + 5s)$

$\Rightarrow x = 31 + 55s \ (\Rightarrow x \equiv 31(\bmod 55))$ ___A1___

Note: Accept other methods *eg* formula, Diophantine equation.

Note: Accept other equivalent answers e.g. $-79(\bmod 55)$.

[3 marks]

b. $41^{82} \equiv 8^{82}(\bmod 11)$

by Fermat's little theorem $8^{10} \equiv 1(\bmod 11) \quad \left(\text{or } 41^{10} \equiv 1(\bmod 11)\right)$ ___M1___

$8^{82} \equiv 8^2(\bmod 11)$ ___M1___

$\equiv 9(\bmod 11)$ ___(A1)___

remainder is 9 ___A1___

Note: Accept simplifications done without Fermat.

[4 marks]

c. $41^{82} \equiv 1^{82} \equiv 1(\bmod 5)$ ___A1___

so 41^{82} has a remainder 1 when divided by 5 and a remainder 9 when divided by 11 ___R1___

hence by part (a) the remainder is 31 ___A1___

[3 marks]

Total [10 marks]

Examiners report

a. A variety of methods were used here. The Chinese Remainder Theorem method (Method 2 on the mark-scheme) is probably the most instructive.

   Candidates who tried to do it by formula often (as usual) made mistakes and got it wrong. Marks were lost by just saying 31 and not giving mod (55).

b. Time was lost here by not using Fermat's Little Theorem as a starting point, although the ad hoc methods will work.

c. Although it said use parts (a) and (b) not enough candidates saw the connection.

Consider an integer a with $(n+1)$ digits written as $a = 10^n a_n + 10^{n-1} a_{n-1} + \ldots + 10 a_1 + a_0$, where $0 \leqslant a_i \leqslant 9$ for $0 \leqslant i \leqslant n$, and $a_n \neq 0$.

(a)   Show that $a \equiv (a_n + a_{n-1} + \ldots + a_0)(\mathrm{mod}\,9)$.

(b)   Hence find all pairs of values of the single digits x and y such that the number $a = 476x212y$ is a multiple of 45.

(c)   (i)   If $b = 34\,390$ in base 10, show that b is $52\,151$ written in base 9.

(ii)   Hence find b^2 in base 9, by performing all the calculations without changing base.

Markscheme

(a) $10 \equiv 1(\bmod\ 9) \Rightarrow 10^i \equiv 1(\mathrm{mod}\,9),\ i = 1,\ \ldots,\ n$ **M1A1**

$\Rightarrow 10^i a_i \equiv a_i(\mathrm{mod}\,9),\ i = 1,\ n$ **M1**

Note: Allow $i = 0$ but do not penalize its omission.

$\Rightarrow (10^n a_n + 10^{n-1} a_{n-1} + \ldots + a_0) \equiv (a_n + a_{n-1} + \ldots + a_0)(\mathrm{mod}\,9)$ **AG**

[3 marks]

(b) $4 + 7 + 6 + x + 2 + 1 + 2 + y = 9k,\ k \in \mathbb{Z}$ **(M1)**

$\Rightarrow (22 + x + y) \equiv 0(\bmod\ 9),\ \Rightarrow (x + y) \equiv 5(\mathrm{mod}\,9)$

$\Rightarrow x + y = 5$ or 14 **A1**

if 5 divides a, then $y = 0$ or 5 **M1**

so $y = 0 \Rightarrow x = 5,\ (ie\ (x,\ y) = (5,\ 0))$ **A1**

$y = 5 \Rightarrow x = 0$ or $x = 9,\ (ie\ (x,\ y) = (0,\ 5)$ or $(x,\ y) = (9,\ 5))$ **A1A1**

[6 marks]

(c)   (i)

34390	1
3821	5
424	1
47	2
5	

(M1)A1

$b = (52\,151)_9$ **AG**

(ii)

				×	5	2	1	5	1
					5	2	1	5	1
					5	2	1	5	1
			2	8	1	7	7	5	
			5	2	1	5	1		
	1	1	4	3	1	2			
2	8	1	7	7	5				
3	0	4	2	3	5	8	1	1	1

M1A3

Note: *M1* for attempt, *A1* for two correct lines of multiplication, *A2* for two correct lines of multiplication and a correct addition, *A3* for all correct.

[6 marks]

Examiners report

Surprisingly few good answers. Part (a) had a number of correct solutions, but there were also many that seemed to be a memorised solution, not properly expressed – and consequently wrong. In part (b) many failed to understand the question, not registering that x and y were digits rather than numbers. Part (c)(i) was generally well answered, although there were a number of longer methods applied, and few managed to do (c)(ii).

(a)   Find the general solution for the following system of congruences.

$\quad N \equiv 3(\bmod 11)$

$\quad N \equiv 4(\bmod 9)$

$\quad N \equiv 0(\bmod 7)$

(b)   Find all values of N such that $2000 \leqslant N \leqslant 4000$.

Markscheme

(a) $N = 3 + 11t$ *M1*

$3 + 11t \equiv 4(\bmod 9)$

$2t \equiv 1(\bmod 9)$ *(A1)*

multiplying by 5, $10t \equiv 5(\bmod 9)$ *(M1)*

$t \equiv 5(\bmod 9)$ *A1*

$t = 5 + 9s$ *M1*

$N = 3 + 11(5 + 9s)$

$N = 58 + 99s$ *A1*

$58 + 99s \equiv 0(\bmod 7)$

$2 + s \equiv 0(\bmod 7)$

$s \equiv 5(\bmod 7)$ *A1*

$s = 5 + 7u$ *M1*

$N = 58 + 99(5 + 7u)$

$N = 553 + 693u$ *A1*

Note: Allow solutions that are done by formula or an exhaustive, systematic listing of possibilities.

[9 marks]

(b) $u = 3$ or 4

hence N = 553 + 2079 = 2632 or N = 553 + 2772 = 3325   *A1A1*

[2 marks]

Total [11 marks]

Examiners report

This was a standard Chinese remainder theorem problem that many candidates gained good marks on. Some candidates employed a formula,

which was fine if they remembered it correctly (but not all did), although it did not always show good understanding of the problem.

Anna is playing with some cars and divides them into three sets of equal size. However, when she tries to divide them into five sets of equal size,

there are four left over. Given that she has fewer than 50 cars, what are the possible numbers of cars she can have?

Markscheme

METHOD 1

let x be the number of cars

we know $x \equiv 0(\bmod 3)$ *(A1)*

also $x \equiv 4(\bmod 5)$ *(A1)*

so $x = 3t \Rightarrow 3t \equiv 4(\bmod 5)$ *M1*

$\Rightarrow 6t \equiv 8(\bmod 5)$

$\Rightarrow t \equiv 3(\bmod 5)$

$\Rightarrow t = 3 + 5s$

$\Rightarrow x = 9 + 15s$ *A1*

since there must be fewer than 50 cars, x = 9, 24, 39   *A1A1A1*

Note: Only award two of the final three *A1* marks if more than three solutions are given.

[7 marks]

METHOD 2

x is a multiple of 3 that ends in 4 or 9   *R4*

therefore x = 9, 24, 39   *A1A1A1*   *N3*

Note: Only award two of the final three *A1* marks if more than three solutions are given.

[7 marks]

Examiners report

There were a number of totally correct solutions to this question, but some students were unable to fully justify their results.

---

a. Consider the integers $a = 871$ and $b = 1157$, given in base 10. [7]

   (i)   Express a and b in base 13.

   (ii)   Hence show that $\gcd(a,\ b) = 13$.

b. A list L contains $n + 1$ distinct positive integers. Prove that at least two members of L leave the same remainder on division by n. [4]

c. Consider the simultaneous equations [6]

     $4x + y + 5z = a$

     $2x + z = b$

     $3x + 2y + 4z = c$

   where $x,\ y,\ z \in \mathbb{Z}$.

   (i)   Show that 7 divides $2a + b - c$.

   (ii)   Given that $a = 3$, $b = 0$ and $c = -1$, find the solution to the system of equations modulo 2.

d. Consider the set P of numbers of the form $n^2 - n + 41,\ n \in \mathbb{N}$. [6]

   (i)   Prove that all elements of P are odd.

   (ii)   List the first six elements of P for $n = 0, 1, 2, 3, 4, 5$.

   (iii) Show that not all elements of P are prime.

Markscheme

a. (i)   **METHOD 1**

using a relevant list of powers of 13: (1), 13, 169, (2197)   ***M1***

$871 = 5 \times 13^2 + 2 \times 13$ ***A1***

$871 = 520_{13}$ ***A1***

$1157 = 6 \times 13^2 + 11 \times 13$ ***A1***

$1157 = 6\text{B}0_{13}$ ***A1***

METHOD 2

attempted repeated division by 13   ***M1***

$871 \div 13 = 67;\ 67 \div 13 = 5\,\text{rem}2$ ***A1***

$871 = 520_{13}$ ***A1***

$1157 \div 13 = 89;\ 89 \div 13 = 6\,\text{rem}11$ ***A1***

$1157 = 6\text{B}0_{13}$ ***A1***

Note: Allow (11) for B only if brackets or equivalent are present.

(ii) $871 = 13 \times 67;\ 1157 = 13 \times 89$ ***(M1)***

67 and 89 are primes (base 10) or they are co-prime   ***A1***

So $\gcd(871,\ 1157) = 13$ ***AG***

Note: Must be done by hence not Euclid's algorithm on 871 and 1157.

[7 marks]

b. let K be the set of possible remainders on division by n ***(M1)***

then $K = \{0, 1, 2, \ldots n - 1\}$ has n members   ***A1***

because $n < n + 1 \ (= n(L))$ ***A1***

by the pigeon-hole principle (appearing anywhere and not necessarily mentioned by name as long as is explained)   ***R1***

at least two members of L correspond to one member of K ***AG***

[4 marks]

c. (i)   form the appropriate linear combination of the equations: *(M1)*

$2a + b - c = 7x + 7z$ ***A1***

$= 7(x + z)$ ***R1***

so 7 divides $2a + b - c$ ***AG***

(ii)   modulo 2, the equations become   *M1*

$y + z = 1$

$z = 0$ ***A1***

$x = 1$

solution: $(1, 1, 0)$ ***A1***

Note: Award full mark to use of GDC (or done manually) to solve the system giving $x = -1$, $y = -3$, $z = 2$ and then converting mod 2.

[6 marks]

d. (i)   separate consideration of even and odd n *M1*

$\text{even}^2 - \text{even} + \text{odd}$ is odd   ***A1***

$\text{odd}^2 - \text{odd} + \text{odd}$ is odd   ***A1***

all elements of P are odd   ***AG***

Note: Allow other methods *eg*, $n^2 - n = n(n - 1)$ which must be even.

(ii)   the list is $[41, 41, 43, 47, 53, 61]$ ***A1***

(iii) $41^2 - 41 + 41 = 41^2$ divisible by 41   ***A1***

but is not a prime   ***R1***

the statement is disproved (by counterexample) ***AG***

[6 marks]

Examiners report

a.   [N/A]

b.   [N/A]

c.   [N/A]

d.   [N/A]

---

a.   One version of Fermat's little theorem states that, under certain conditions, $a^{p-1} \equiv 1(\bmod\ p)$.   [6]

   (i)   Show that this result is not true when $a = 2$, $p = 9$ and state which of the conditions is not satisfied.

   (ii)   Find the smallest positive value of k satisfying the congruence $2^{45} \equiv k(\bmod\ 9)$.

b.   Find all the integers between 100 and 200 satisfying the simultaneous congruences $3x \equiv 4(\bmod\ 5)$ and $5x \equiv 6(\bmod\ 7)$.   [6]

Markscheme

a. (i) $2^8 = 256 \equiv 4(\bmod\ 9)$ (so not true)   ***A1***

9 is not prime   *A1*

(ii)   consider various powers of 2, *e.g.* obtaining   *M1*

$2^6 = 64 \equiv 1(\bmod 9)$ *A1*

therefore

$2^{45} = (2^6)^7 \times 2^3$ *M1*

$\equiv 8(\bmod 9)$ (so $k = 8$)   *A1*

[6 marks]

b. **EITHER**

the solutions to $3x \equiv 4(\bmod 5)$ are 3, 8, 13, 18, 23,…   *M1A1*

the solutions to $5x \equiv 6(\bmod 7)$ are 4, 11, 18,…   *A1*

18 is therefore the smallest solution   *A1*

the general solution is

$18 + 35n$, $n \in \mathbb{Z}$ *M1*

the required solutions are therefore 123, 158, 193   *A1*

OR

$3x \equiv 4(\bmod 5) \Rightarrow 2 \times 3x \equiv 2 \times 4(\bmod 5) \Rightarrow x \equiv 3(\bmod 5)$ *A1*

$\Rightarrow x = 3 + 5t$ *M1*

$\Rightarrow 15 + 25t \equiv 6(\bmod 7) \Rightarrow 4t \equiv 5(\bmod 7) \Rightarrow 2 \times 4t \equiv 2 \times 5(\bmod 7) \Rightarrow t \equiv 3(\bmod 7)$ *A1*

$\Rightarrow t = 3 + 7n$ *A1*

$\Rightarrow x = 3 + 5(3 + 7n) = 18 + 35n$ *M1*

the required solutions are therefore 123, 158, 193   *A1*

OR

using the Chinese remainder theorem formula method

first convert the congruences to $x \equiv 3(\bmod 5)$ and $x \equiv 4(\bmod 7)$ *A1A1*

$M = 35, M_1 = 7, M_2 = 5, m_1 = 5, m_2 = 7, a_1 = 3, a_2 = 4$

x_1 is the solution of $M_2x_2 \equiv 1(\bmod m_1)$, *i.e.* $7x_1 \equiv 1(\bmod 5)$ so $x_1 = 3$

x_2 is the solution of $M_2x_2 \equiv 1(\bmod m_2)$, *i.e.* $5x_2 \equiv 1(\bmod 7)$ so $x_2 = 3$

a solution is therefore

$x = a_1M_1x_1 + a_2M_2x_2$ *M1*

$= 3 \times 7 \times 3 + 4 \times 5 \times 3 = 123$ *A1*

the general solution is $123 + 35n$, $n \in \mathbb{Z}$ *M1*

the required solutions are therefore 123, 158, 193   *A1*

[6 marks]

Examiners report

A simple connected planar graph, has e edges, v vertices and f faces.

a. (i) Show that $2e \geq 3f$ if $v > 2$. [6]

   (ii) Hence show that K_5, the complete graph on five vertices, is not planar.

b. (i) State the handshaking lemma. [4]

   (ii) Determine the value of f, if each vertex has degree 2.

c. Draw an example of a simple connected planar graph on 6 vertices each of degree 3. [2]

Markscheme

a. (i) **METHOD 1**

   attempting to use $f = e - v + 2$ and $e \leq 3v - 6$ (if $v > 2$) **(M1)**

   $2e \leq 6v - 12 = 6(e - f + 2) - 12$ **M1A1**

   leading to $2e \geq 3f$ **AG**

   METHOD 2

   each face is bounded by at least three edges **A1**

   Note: Award **A1** for stating $e \geq 3f$.

   each edge either separates two faces or, if an edge is interior to a face, it gets counted twice **R1**

   Note: Award **R1** for stating that each edge contributes two to the sum of the degrees of the faces (or equivalent) *ie*, $\sum \deg(F) = 2e$.

   adding up the edges around each face **R1**

   leading to $2e \geq 3f$ **AG**

   (ii) K_5 has $e = 10$ **A1**

   if the graph is planar, $f = 7$ **A1**

   this contradicts the inequality obtained above **R1**

   hence the graph is non-planar **AG**

   [6 marks]

b. (i) the sum of the vertex degrees $= 2e$ (or is even) or equivalent **A1**
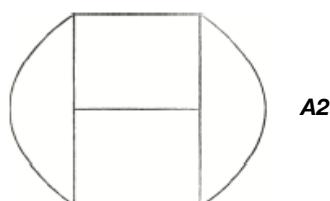
   (ii) if each vertex has degree 2, then $2v = 2e$ **A1**

   substituting $v = e$ into Euler's formula **M1**

   $f = 2$ **A1**

   [4 marks]

c. for example,

   A2

   [2 marks]

Examiners report
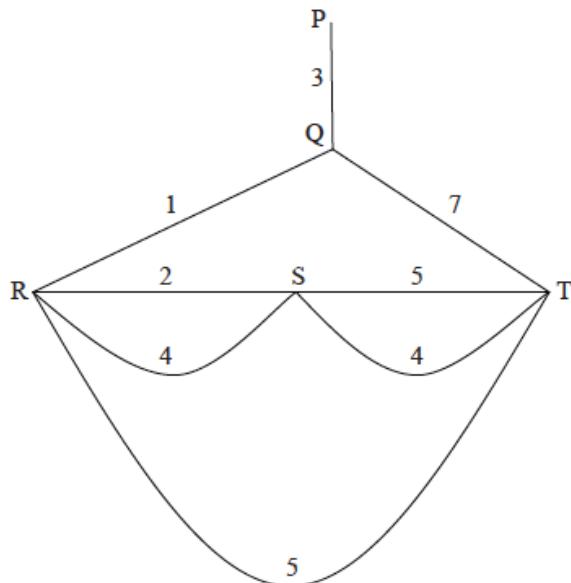
a. In part (a) (i), many candidates tried to prove $2e \geq 3f$ with numerical examples. Only a few candidates were able to prove this inequality correctly.

   In part (a) (ii), most candidates knew that K_5 has 10 edges. However, a number of candidates simply drew a diagram with any number of faces and used this particular representation as a basis for their 'proof'. Many candidates did not recognise the 'hence' requirement in part (a) (ii).

b. In part (b) (i), many candidates stated the 'handshaking lemma' incorrectly by relating it to the 'handshake problem'. In part (b) (ii), only a few candidates determined that $v = e$ and hence found that $f = 2$.

c. In part (c), a reasonable number of candidates were able to draw a simple connected planar graph on 6 vertices each of degree 3. The most common error here was to produce a graph that contained a multiple edge(s).

---

a. The weights of the edges of a graph H are given in the following table. [8]

	A	B	C	D	E	F	G
A	–	5	4	–	–	–	–
B	5	–	–	–	5	–	–
C	4	–	–	5	2	–	–
D	–	–	5	–	3	–	6
E	–	5	2	3	–	5	4
F	–	–	–	–	5	–	1
G	–	–	–	6	4	1	–

   (i)   Draw the weighted graph H.

   (ii)   Use Kruskal's algorithm to find the minimum spanning tree of H. Your solution should indicate the order in which the edges are added.

   (iii)   State the weight of the minimum spanning tree.

b. Consider the following weighted graph. [3]

(i) Write down a solution to the Chinese postman problem for this graph.

(ii) Calculate the total weight of the solution.
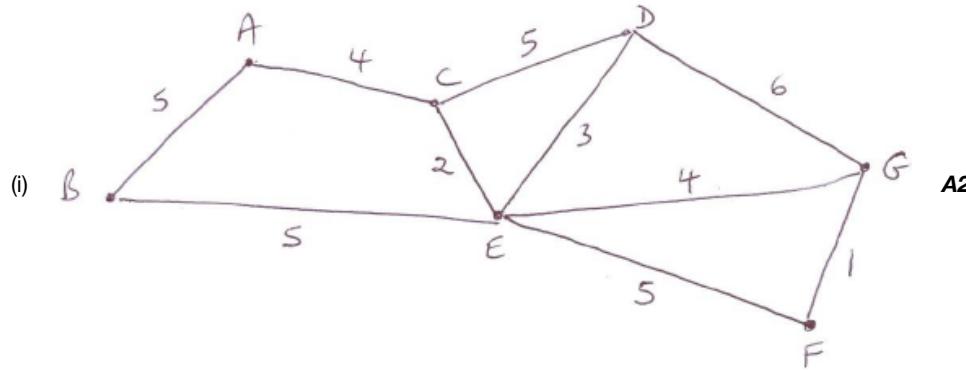
c. (i)   State the travelling salesman problem.   [3]

(ii)   Explain why there is no solution to the travelling salesman problem for this graph.

Markscheme

a.

(i)   *B*

A2

Note: Award **A1** if one edge is missing. Award **A1** if the edge weights are not labelled.

(ii)   the edges are added in the order:

$FG1$ **M1A1**

$CE2$ **A1**

$ED3$

$EG4$ **A1**

$AC4$

Note: EG and AC can be added in either order.

(Reject EF)

(Reject CD)

$EB5$ **OR** $AB5$ **A1**

Notes: The minimum spanning tree does not have to be seen.

If only a tree is seen, the order by which edges are added must be clearly indicated.

(iii)   19   **A1**

[8 marks]

b. (i)   eg, $PQRSRTSTQP$ **OR** $PQTSTRSRQP$ **M1A1**

Note: Award **M1** if in either (i) or (ii), it is recognised that edge PQ is needed twice.

(ii)   total weight $= 34$ **A1**

[3 marks]

c. (i)   to determine a cycle where each vertex is visited once only (Hamiltonian cycle)   **A1**

of least total weight   **A1**

(ii)   **EITHER**

to reach P, Q must be visited twice which contradicts the definition of the TSP **R1**

OR

the graph is not a complete graph and hence there is no solution to the TSP **R1**

[3 marks]

Total [14 marks]

Examiners report

a. Part (a) was generally very well answered. Most candidates were able to correctly sketch the graph of H and apply Kruskal's algorithm to determine the minimum spanning tree of H. A few candidates used Prim's algorithm (which is no longer part of the syllabus).

b. Most candidates understood the Chinese Postman Problem in part (b) and knew to add the weight of PQ to the total weight of H. Some candidates, however, did not specify a solution to the Chinese Postman Problem while other candidates missed the fact that a return to the initial vertex is required.

c. In part (c), many candidates had trouble succinctly stating the Travelling Salesman Problem. Many candidates used an 'edge' argument rather than simply stating that the Travelling Salesman Problem could not be solved because to reach vertex P, vertex Q had to be visited twice.

---

a. Show that there are exactly two solutions to the equation $1982 = 36a + 74b$, with $a,\ b \in \mathbb{N}$.   [8]

b. Hence, or otherwise, find the remainder when 1982^{1982} is divided by 37.   [5]

Markscheme

a. $74 = 2 \times 36 + 2$ **OR** $\gcd(36,\ 74) = 2$ *(A1)*

$2 = (-2)(36) + (1)(74)$ *M1*

$1982 = (-1982)(36) + (991)(74)$ *A1*

so solutions are $a = -1982 + 37t$ and $b = 991 - 18t$ *M1A1*

$a,\ b \in \mathbb{N}$ so $\frac{1982}{37} \leq t \leq \frac{991}{18}$ $(15.56\ldots \leq t \leq 55.055\ldots)$ *(M1)(A1)*

t can take values 54 or 55 only   *A1AG*

(Or the solutions are $(16,\ 19)$ or $(53,\ 1)$)

Note: Accept arguments from one solution of increasing/decreasing a by 37 and increasing/decreasing b by 18 to give the only possible positive solutions.

[8 marks]

b. $1982 = 53 \times 36 + 74$

$1982 = 55 \times 36 + 2$ *(M1)*

$1982^{36} \equiv 1(\mod 37) \, (\text{by FLT})$ *(M1)*

$1982^{1982} = 1982^{36 \times 55 + 2} \equiv 1982^2 (\mod 37)$ *(A1)*

$\equiv 34(\mod 37)$ *A1*

so the remainder is 34 *A1*

Note: 1982 in the base can be replaced by 21.

Note: Award the first *(M1)* if the 1982 in the experiment is correctly broken down to a smaller number.

[5 marks]

Total [13 marks]

Examiners report

a. [N/A]
b. [N/A]

(a)   Using Fermat's little theorem, show that, in base 10, the last digit of n is always equal to the last digit of n^5 .

(b)   Show that this result is also true in base 30.

Markscheme

(a)   using Fermat's little theorem $n^5 \equiv n(\mod 5)$ *(M1)*

$n^5 - n \equiv 0(\mod 5)$ *A1*

now $n^5 - n = n(n^4 - 1)$ *(M1)*

$= n(n^2 - 1)(n^2 + 1)$

$= n(n - 1)(n + 1)(n^2 + 1)$ *A1*

hence one of the first two factors must be even   *R1*

i.e. $n^5 - n \equiv 0(\mod 2)$

thus $n^5 - n$ is divisible by 5 and 2

hence it is divisible by 10   *R1*

in base 10, since $n^5 - n$ is divisible by 10, then $n^5 - n$ must end in zero and hence n^5 and n must end with the same digit   *R1*

[7 marks]

(b)   consider $n^5 - n = n(n - 1)(n + 1)(n^2 + 1)$

this is divisible by 3 since the first three factors are consecutive integers   *R1*

hence $n^5 - n$ is divisible by 3, 5 and 2 and therefore divisible by 30

in base 30, since $n^5 - n$ is divisible by 30, then $n^5 - n$ must end in zero and hence n^5 and n must end with the same digit   *R1*

[2 marks]

Total [9 marks]

Examiners report

There were very few fully correct answers. If Fermat's little theorem was known, it was not well applied.

---

a. Write 57128 as a product of primes. [4]

c. Prove that $22|\,5^{11} + 17^{11}$. [4]

Markscheme

a. $457\,128 = 2 \times 228\,564$

$228\,564 = 2 \times 114\,282$
$114\,282 = 2 \times 57\,141$
$57\,141 = 3 \times 19\,047$
$19\,047 = 3 \times 6349$
$6349 = 7 \times 907$ ***M1A1***
trial division by 11, 13, 17, 19, 23 and 29 shows that 907 is prime   ***R1***
therefore $457\,128 = 2^3 \times 3^2 \times 7 \times 907$ ***A1***
[4 marks]

c. by a corollary to Fermat's Last Theorem

$5^{11} \equiv 5(\bmod\ 11)$ and $17^{11} \equiv 17(\bmod\ 11)$ ***M1A1***

$5^{11} + 17^{11} \equiv 5 + 17 \equiv 0(\bmod\ 11)$ ***A1***
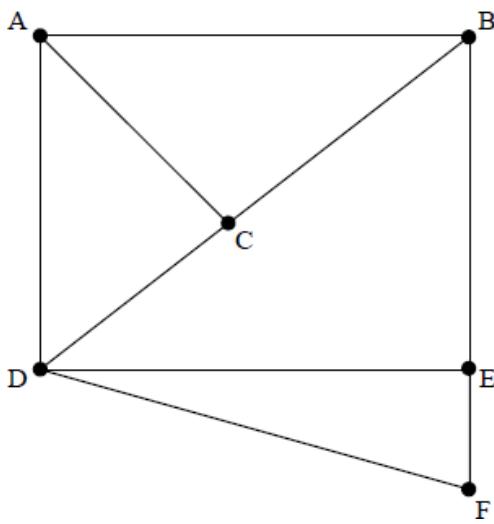
this combined with the evenness of LHS implies $25|\,5^{11} + 17^{11}$ ***R1AG***

[4 marks]

Examiners report

a. Some candidates were obviously not sure what was meant by 'product of primes' which surprised the examiner.

c. There were some reasonable attempts at part (c) using powers rather than Fermat's little theorem.

The diagram below shows the graph G with vertices A, B, C, D, E and F.

(i) Determine if any Hamiltonian cycles exist in G. If so, write one down.

Otherwise, explain what feature of G makes it impossible for a Hamiltonian cycle to exist.

(ii) Determine if any Eulerian circuits exist in G. If so, write one down.

Otherwise, explain what feature of G makes it impossible for an Eulerian circuit to exist.

Markscheme

(i) any Hamiltonian circuit ACBEFDA *A2*
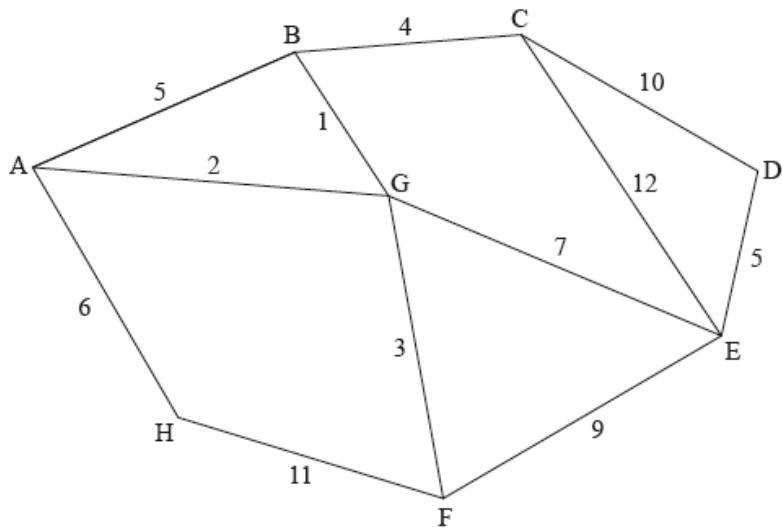
(ii) no Eulerian circuit exists because the graph contains vertices of odd degree *A2*

[4 marks]

Examiners report

Parts (a) and (b) were well answered by many candidates. In (c), candidates who tried to prove the result by adding edges to a drawing of G were given no credit. Candidates should be aware that the use of the word 'Prove' indicates that a formal treatment is required Solutions to (d) were often disappointing although a graphical solution was allowed here.

The following diagram shows a weighted graph.

(a)   Use Kruskal's algorithm to find a minimum spanning tree, clearly showing the order in which the edges are added.

(b)   Sketch the minimum spanning tree found, and write down its weight.

Markscheme

(a)   use Kruskal's algorithm: begin by choosing the shortest edge and then select a sequence of edges of non-decreasing weights, checking at

each stage that no cycle is completed   *(M1)*

choice edge weight
1   BG   1
2   AG   2
3   FG   3
4   BC   4
5   DE   5
6   AH   6
7   EG   7

A1
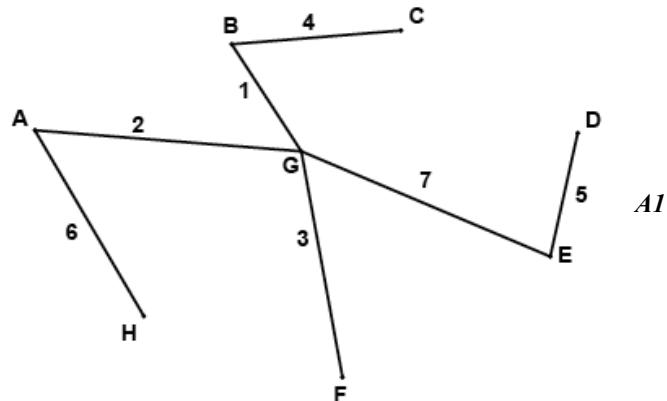
A3

Note: *A1* for steps 2–4, *A1* for step 5 and *A1* for steps 6, 7.
     Award marks only if it is clear that Kruskal's algorithm is being used.

[5 marks]

(b)   weight $= 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ *A1*

A1

Note: Award *FT* only if it is a spanning tree.

[2 marks]

Examiners report

Well answered by the majority of candidates.

Let G be a simple, connected, planar graph.

(a) (i) Show that Euler's relation $f - e + v = 2$ is valid for a spanning tree of G.

(ii) By considering the effect of adding an edge on the values of f, e and v show that Euler's relation remains true.

(b) Show that K_5 is not planar.

Markscheme

(a) (i) A spanning tree with v vertices and $(v-1)$ edges where $f = 1$ *A1A1*

$f - e + v = 1 - (v - 1) + v = 2$ *M1*

So the formula is true for the tree   *AG*

(ii) Adding one edge connects two different vertices, and hence an extra face is created   *M1R1*

This leaves v unchanged but increases both e and f by 1 leaving $f - e + v$ unchanged. Hence $f - e + v = 2$.   *R1R1*

[7 marks]

(b) Using $e \leqslant 3v - 6$,   *M1*

for K_5, $v = 5$ and $e = \binom{5}{2} = 10$ *A1A1*

but $3v - 6 = 3(5) - 6 = 9$ *A1*

9 is not greater or equal to 10 so K_5 is not planar   *R1*

[5 marks]

Total [12 marks]

Examiners report

Part (a)(i) was done successfully but many students did not read part(ii) carefully. It said **'adding an edge'** nothing else. Many candidates assumed it was necessary to add a vertex when this was not the case.

Part (b) was not found to be beyond many candidates if they used the inequality $e \leqslant 3v - 6$

(a) Write down Fermat's little theorem.

(b) In base 5 the representation of a natural number X is $\left(k00013(5-k)\right)_5$.

This means that $X = k \times 5^6 + 1 \times 5^2 + 3 \times 5 + (5 - k)$.

In base 7 the representation of X is $\left(a_n a_{n-1} \ldots a_2 a_1 a_0\right)_7$.

Find a_0.

(c) Given that $k = 2$, find X in base 7.

Markscheme

(a) **EITHER**

if p is a prime $a^p \equiv a(\bmod\ p)$ *A1A1*

OR

if p is a prime and $a \not\equiv 0(\bmod\ p)$ then $a^{p-1} \equiv 1(\bmod\ p)$ *A1A1*

Note: Award *A1* for p being prime and *A1* for the congruence.

[2 marks]

(b) $a_0 \equiv X(\bmod\ 7)$ *M1*

$X = k \times 5^6 + 25 + 15 + 5 - k$

by Fermat $5^6 \equiv 1(\bmod\ 7)$ *R1*

$X = k + 45 - k(\bmod\ 7)$ *(M1)*

$X = 3(\bmod\ 7)$ *A1*

$a_0 = 3$ *A1*

[5 marks]

(c) $X = 2 \times 5^6 + 25 + 15 + 3 = 31\,293$ *A1*

EITHER

$X - 7^5 = 14\,486$ *(M1)*

$X - 7^5 - 6 \times 7^4 = 80$

$X - 7^5 - 6 \times 7^4 - 7^2 = 31$

$X - 7^5 - 6 \times 7^4 - 7^2 - 4 \times 7 = 3$

$X = 7^5 + 6 \times 7^4 + 7^2 + 4 \times 7 + 3$ *(A1)*

$X = (160\,143)_7$ *A1*

OR

$31\,293 = 7 \times 4470 + 3$ *(M1)*

$4470 = 7 \times 638 + 4$

$638 = 7 \times 91 + 1$

$91 = 7 \times 13 + 0$

$13 = 7 \times 1 + 6$ *(A1)*

$X = (160\,143)_7$ *A1*

[4 marks]

Total [11 marks]

Examiners report

Fermat's little theorem was reasonably well known. Not all candidates took the hint to use this in the next part and this part was not done well.

Part (c) could and was done even if part (b) was not.

(a)   Show that, for a connected planar graph,

$$v + f - e = 2.$$

(b)   Assuming that $v \geqslant 3$, explain why, for a simple connected planar graph, $3f \leqslant 2e$ and hence deduce that $e \leqslant 3v - 6$.

(c)   The graph G and its complement G' are simple connected graphs, each having 12 vertices. Show that G and G' cannot both be planar.

Markscheme

(a)   start with a graph consisting of just a single vertex   *M1*

for this graph, $v = 1, f = 1$ and $e = 0$, the relation is satisfied   *A1*

Note: Allow solutions that begin with 2 vertices and 1 edge.

to extend the graph you either join an existing vertex to another existing vertex which increases e by 1 and f by 1 so that $v + f - e$ remains equal to

2   *M1A1*

or add a new vertex and a corresponding edge which increases e by 1 and v by 1 so that $v + f - e$ remains equal to 2   *M1A1*

therefore, however we build up the graph, the relation remains valid   *R1*

[7 marks]

(b)   since every face is bounded by at least 3 edges, the result follows by counting up the edges around each face   *R1*

the factor 2 follows from the fact that every edge bounds (at most) 2 faces   *R1*

hence $3f \leqslant 2e$ *AG*

from the Euler relation, $3f = 6 + 3e - 3v$ *M1*

substitute in the inequality, $6 + 3e - 3v \leqslant 2e$ *A1*

hence $e \leqslant 3v - 6$ *AG*

[4 marks]

(c)   let G have e edges   *M1*

since G and G' have a total of $\begin{pmatrix} 12 \\ 2 \end{pmatrix} = 66$ edges   *A1*

it follows that G' has $66 - e$ edges   *A1*

for planarity we require

$e \leqslant 3 \times 12 - 6 = 30$ *M1A1*

and $66 - e \leqslant 30 \Rightarrow e \geqslant 36$ *A1*

these two inequalities cannot both be met indicating that both graphs cannot be planar   *R1*

[7 marks]

Total [18 marks]

Examiners report

Parts (a) and (b) were found difficult by many candidates with explanations often inadequate. In (c), candidates who realised that the union of a graph with its complement results in a complete graph were often successful.

State Fermat's little theorem.

Markscheme

if p is a prime (and $a \equiv 0(\bmod\ p)$ with $a \in \mathbb{Z}$) then   *A1*
$a^{p-1} \equiv 1(\bmod\ p)$ *A1*
[2 marks]
Note: Accept $a^p \equiv a(\bmod\ p)$.

Examiners report

Fermat's little theorem was reasonably well known. Some candidates forgot to mention that p was a prime. Not all candidates took the hint to use this in the next part.

The positive integer p is an odd prime number.

a. Show that $\displaystyle\sum_{k=1}^{p} k^p \equiv 0(\bmod\ p)$. [4]

b. Given that $\displaystyle\sum_{k=1}^{p} k^{p-1} \equiv n(\bmod\ p)$ where $0 \leqslant n \leqslant p-1$, find the value of n. [4]

Markscheme

a. using Fermat's little theorem,

$k^p \equiv k(\bmod\ p)$ *(M1)*

therefore,

$\displaystyle\sum_{k=1}^{p} k^p \equiv \sum_{k=1}^{p} k(\bmod\ p)$ *M1*

$\equiv \frac{p(p+1)}{2}(\bmod\ p)$ *A1*

$\equiv 0(\bmod\ p)$ *AG*

since $\frac{(p+1)}{2}$ is an integer (so that the right-hand side is a multiple of p)   *R1*

[4 marks]

b. using the alternative form of Fermat's little theorem,

$k^{p-1} \equiv 1(\bmod\ p),\ 1 \leqslant k \leqslant p-1$ *A1*

$k^{p-1} \equiv 0(\bmod\ p),\ k = p$ *A1*

therefore,

$\displaystyle\sum_{k=1}^{p} k^{p-1} \equiv \sum_{k=1}^{p-1} 1\ (+0)(\bmod\ p)$ *M1*

$\equiv p - 1(\bmod\ p)$ *A1*

(so $n = p - 1$)

Note: Allow first *A1* even if qualification on k is not given.

[4 marks]

Examiners report

a. Only the top candidates were able to produce logically, well thought-out proofs. Too many candidates struggled with the summation notation and were not able to apply Fermat's little theorem. There was poor logic i.e. looking at a particular example and poor algebra.

b. Only the top candidates were able to produce logically, well thought-out proofs. Too many candidates struggled with the summation notation and were not able to apply Fermat's little theorem. There was poor logic i.e. looking at a particular example and poor algebra.

(a)   Use the Euclidean algorithm to find the gcd of 324 and 129.

(b)   Hence show that $324x + 129y = 12$ has a solution and find both a particular solution and the general solution.

(c)   Show that there are no integers x and y such that $82x + 140y = 3$.

Markscheme

(a) $324 = 2 \times 129 + 66$ *M1*

$129 = 1 \times 66 + 63$

$66 = 1 \times 63 + 3$ *A1*

hence gcd (324, 129) = 3   *A1*

[3 marks]

(b)   **METHOD 1**
Since $3 \mid 12$ the equation has a solution   *M1*
$3 = 1 \times 66 - 1 \times 63$ *M1*
$3 = -1 \times 129 + 2 \times 66$
$3 = 2 \times (324 - 2 \times 129) - 129$
$3 = 2 \times 324 - 5 \times 129$ *A1*
$12 = 8 \times 324 - 20 \times 129$ *A1*
$(x, y) = (8, -20)$ is a particular solution   *A1*
Note: A calculator solution may gain *M1M1A0A0A1*.

A general solution is $x = 8 + \frac{129}{3}t = 8 + 43t$, $y = -20 - 108t$, $t \in \mathbb{Z}$ *A1*
METHOD 2
$324x + 129y = 12$
$108x + 43y = 4$ *A1*
$108x \equiv 4(\bmod 43) \Rightarrow 27x \equiv 1(\bmod 43)$ *A1*
$x = 8 + 43t$ *A1*
$108(8 + 43t) + 43y = 4$ *M1*
$864 + 4644t + 43y = 4$
$43y = -860 - 4644t$
$y = -20 - 108t$ *A1*
a particular solution (for example $t = 0$) is $(x, y) = (8, -20)$ *A1*
[6 marks]

(c)   **EITHER**
The left side is even and the right side is odd so there are no solutions   *M1R1AG*
[2 marks]
OR
$\gcd(82, 140) = 2$ *A1*
2 does not divide 3 therefore no solutions   *R1AG*
[2 marks]

Total [11 marks]

Examiners report

This problem was not difficult but presenting a clear solution and doing part (b) alongside part (a) in two columns was. The simple answer to part

(c) was often overlooked.

The weights of the edges of a graph G with vertices A, B, C, D and E are given in the following table.

	A	B	C	D	E
A	–	11	18	12	9
B	11	–	17	13	14
C	18	17	–	16	10
D	12	13	16	–	15
E	9	14	10	15	–

a. Starting at A, use the nearest neighbour algorithm to find an upper bound for the travelling salesman problem for G. [4]

b. (i) Use Kruskal's algorithm to find and draw a minimum spanning tree for the subgraph obtained by removing the vertex A from G. [8]

(ii) Hence use the deleted vertex algorithm to find a lower bound for the travelling salesman problem for G.

Markscheme

a. using the nearest neighbour algorithm, starting with A,

A → E, E → C *A1*
C → D, D → B *A1*
B → A *A1*
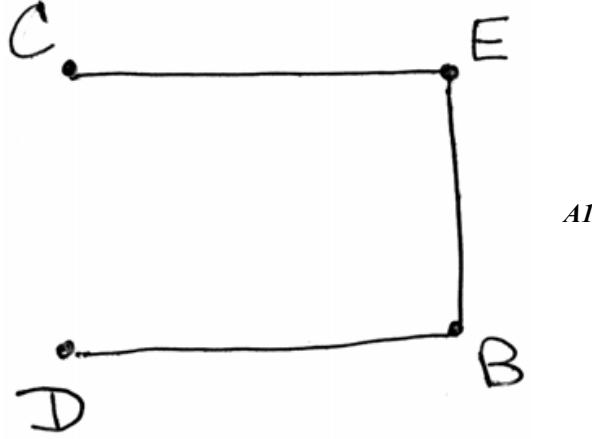
the upper bound is therefore $9 + 10 + 16 + 13 + 11 = 59$ *A1*

[4 marks]

b. (i) the edges are added in the order CE *A1*

BD *A1*
BE *A1*

A1

(ii) the weight of the minimum spanning tree is 37 *(A1)*

we now reconnect A with the 2 edges of least weight *(M1)*

i.e. AE and AB *A1*

the lower bound is therefore $37 + 9 + 11 = 57$ *A1*

[8 marks]

Examiners report
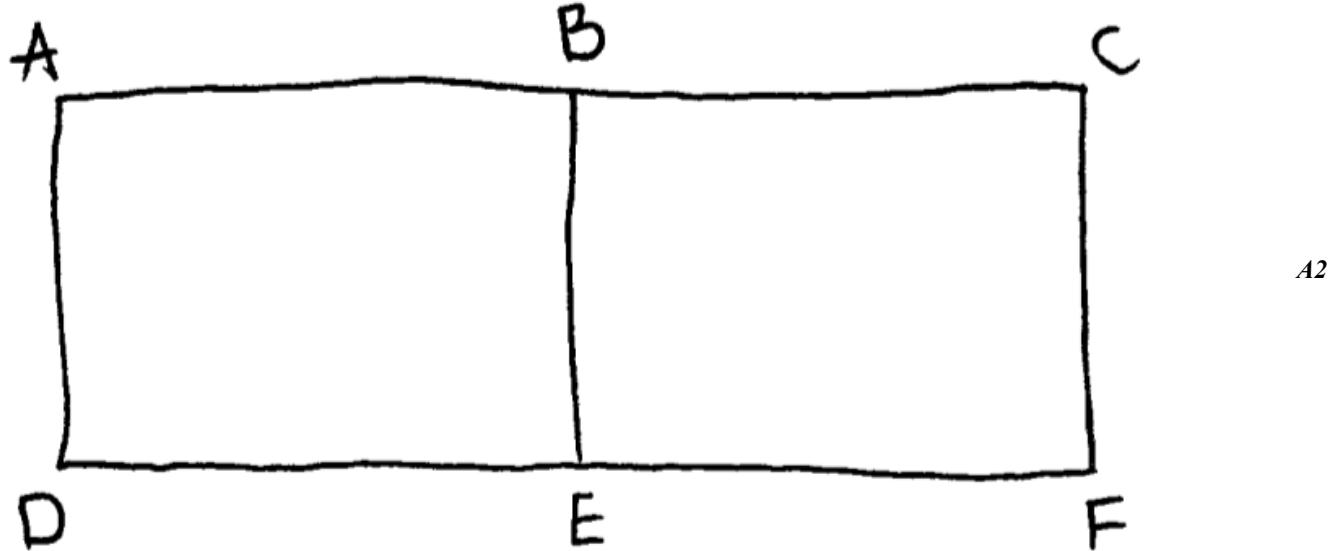
The graph G has adjacency matrix M given below.

$$
\begin{array}{c}
\begin{array}{cccccc} A & B & C & D & E & F \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

a. Draw the graph G . [2]

b. What information about G is contained in the diagonal elements of M^2 ? [1]

d. List the trails of length 4 starting at A and ending at C. [3]

Markscheme

a.

Note: Award *A1* if only one error, *A0* for two or more.

[2 marks]

b. the (k, k) element of M^2 is the number of vertices directly connected to vertex k *A1*

Note: Accept comment about the number of walks of length 2, in which the initial and final vertices coincide.

[1 mark]

d. the trails of length 4 are ABEDC, AFEDC, AFEBC *A1A1A1*

Note: *A1A1A1* for three correct with no additions; *A1A1A0* for all correct, but with additions; *A1A0A0* for two correct with or without additions.

[3 marks]

Examiners report

a. Parts (a) and (c) were generally correctly answered. In part (b), a minority of candidates failed to mention that the starting and end points had

to coincide. A large number of candidates gave all walks (trails were asked for) – an unnecessary loss of marks.

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d. Parts (a) and (c) were generally correctly answered. In part (b), a minority of candidates failed to mention that the starting and end points had to coincide. A large number of candidates gave all walks (trails were asked for) – an unnecessary loss of marks.

Show that a graph is bipartite if and only if it contains only cycles of even length.

Markscheme

Suppose the graph is bipartite so that the vertices belong to one of two disjoint sets M, N.   *M1*

Then consider any vertex V in M. To generate a cycle returning to V, we must go to a vertex in N, then to a vertex in M, then to a vertex in N, then to a vertex in M, *etc.*   *R1*

To return to V, therefore, which belongs to M, an even number of steps will be required.   *R1*

Now suppose the graph contains only cycles of even length.   *M1*

Starting at any vertex V, define the set M as containing those vertices accessible from V in an even number of steps and the set N as containing those vertices accessible from V in an odd number of steps.   *R1*

Suppose that the vertex X belongs to both M and N. Then consider the closed walk from V to X one way and back to V the other way. This closed walk will be of odd length. This closed walk can be contracted to a cycle which will also be of odd length, giving a contradiction to the initial assumption.   *R1*

There can therefore be no vertices common to M and N which shows that the vertices can be divided into two disjoint sets and the graph is bipartite.   *R1*

Consider any edge joining P to vertex Q. Then either P ∈ M in which case Q ∈ M or vice versa. In either case an edge always joins a vertex in M to a vertex in N so the graph is bipartite.   *R1*
[8 marks]
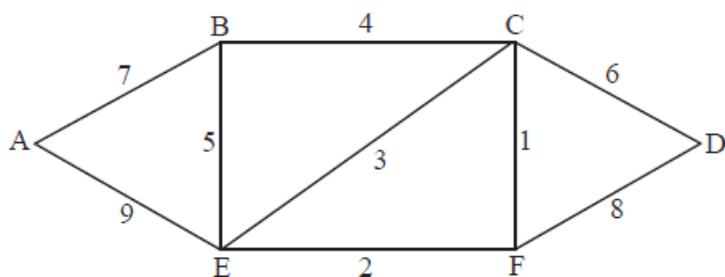
Examiners report

Many candidates made a reasonable attempt at showing that bipartite implies cycles of even length but few candidates even attempted the converse.

---

a. The weighted graph H is shown below.   [6]
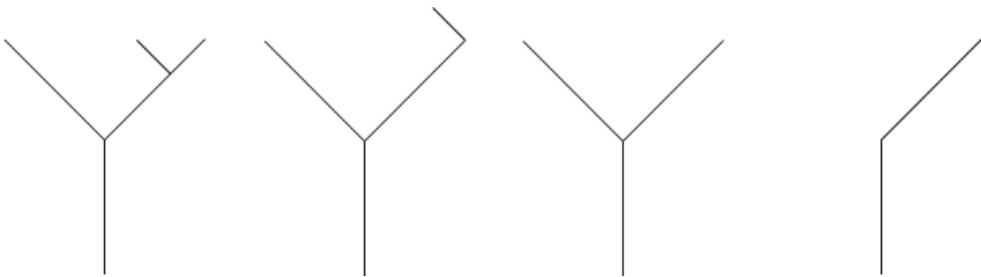
Use Kruskal's Algorithm, indicating the order in which the edges are added, to find and draw the minimum spanning tree for H.

b. (i) A tree has v vertices. State the number of edges in the tree, justifying your answer. [5]

(ii) We will call a graph with v vertices a "forest" if it consists of c components each of which is a tree.
Here is an example of a forest with 4 components.

How many edges will a forest with v vertices and c components have?

Markscheme

a. The edges are included in the order
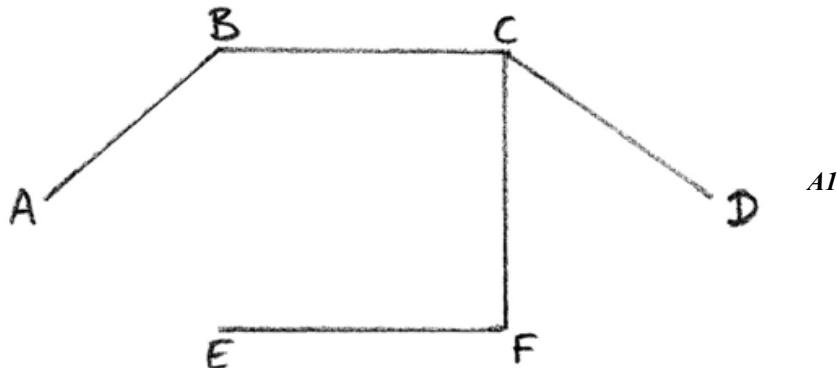
CF   *A1*
EF   *A1*
BC   *A1*
CD   *A1*
AB   *A1*

A1

[6 marks]

b. (i) A tree with v vertices has $v - 1$ edges.   *A1*

Using $v + f = e + 2$ with $f = 1$, the result follows.   *R1*

(ii) Each of the c trees will have one less edge than the number of vertices.   *R1*
Thus the forest will have $v - c$ edges.   *A2*
[5 marks]

Examiners report

a. In (a), many candidates derived the minimum spanning tree although in some cases the method was not clearly indicated as required and some

candidates used an incorrect algorithm. Part (b) was reasonably answered by many candidates although some justifications were unsatisfactory.

Part (c) caused problems for many candidates who found difficulty in writing down a rigorous proof of the required result.

b. In (a), many candidates derived the minimum spanning tree although in some cases the method was not clearly indicated as required and some candidates used an incorrect algorithm. Part (b) was reasonably answered by many candidates although some justifications were unsatisfactory. Part (c) caused problems for many candidates who found difficulty in writing down a rigorous proof of the required result.
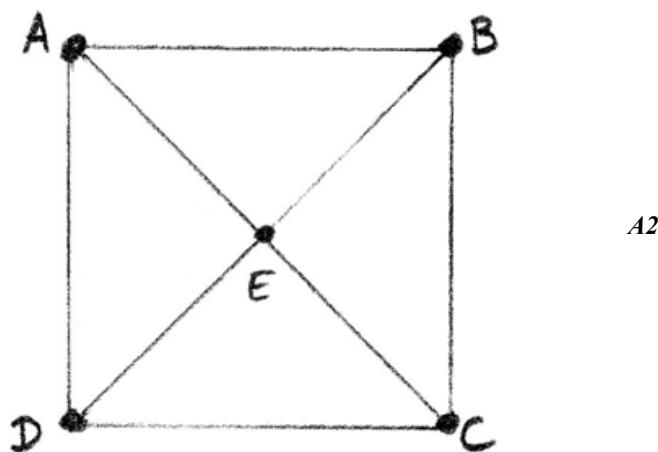
The graph G has the following cost adjacency table.

	A	B	C	D	E
A	-	9	-	8	4
B	9	-	7	-	2
C	-	7	-	7	3
D	8	-	7	-	5
E	4	2	3	5	-

a. Draw G in a planar form. [2]

b. Giving a reason, determine the maximum number of edges that could be added to G while keeping the graph both simple and planar. [4]

c. List all the distinct Hamiltonian cycles in G beginning and ending at A, noting that two cycles each of which is the reverse of the other are [10] to be regarded as identical. Hence determine the Hamiltonian cycle of least weight.

Markscheme

a.

A2

[2 marks]

b. For a simple planar graph containing triangles, $e \leqslant 3v - 6$ *M1*

Here $v = 5$, so $e \leqslant 9$.   *A1*
There are already 8 edges so the maximum number of edges that could be added is 1.   *R1*
This can be done *e.g.* AC or BD   *R1*
[4 marks]

c. The distinct Hamiltonian cycles are

ABCDEA   *A2*
ABCEDA   *A2*
ABECDA   *A2*

AEBCDA   *A2*

Note: Do not penalise extra cycles.

The weights are 32, 32, 29, 28 respectively.   *A1*
The Hamiltonian cycle of least weight is AEBCDA.   *R1*
[10 marks]

Examiners report

a. A fairly common error in (a) was to draw a non-planar version of *G*, for which no credit was given. In (b), most candidates realised that only

   one extra edge could be added but a convincing justification was often not provided. Most candidates were reasonably successful in (c)

   although in some cases not all possible Hamiltonian cycles were stated.

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   one extra edge could be added but a convincing justification was often not provided. Most candidates were reasonably successful in (c)

   although in some cases not all possible Hamiltonian cycles were stated.

The complete graph *H* has the following cost adjacency matrix.
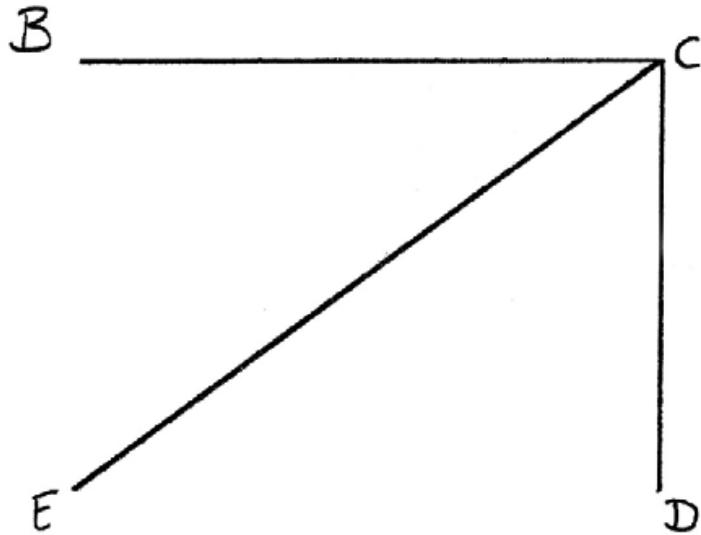
	A	B	C	D	E
A	–	19	17	10	15
B	19	–	11	16	13
C	17	11	–	14	13
D	10	16	14	–	18
E	15	13	13	18	–

Consider the travelling salesman problem for *H* .

a. By first finding a minimum spanning tree on the subgraph of *H* formed by deleting vertex A and all edges connected to A, find a lower   [5]

   bound for this problem.

b. Find the total weight of the cycle ADCBEA.   [1]

c. What do you conclude from your results?   [1]

Markscheme

a. using any method, the minimum spanning tree is   *(M1)*

A2

Note: Accept MST = {BC, EC, DC} or {BC, EB, DC}

Note: In graph, line CE may be replaced by BE.

lower bound = weight of minimum spanning tree + 2 smallest weights connected to A   *(M1)*

= 11 + 13 + 14 + 10 + 15 = 63   *A1*

[5 marks]

b. weight of ADCBEA = 10 + 14 + 11 + 13 + 15 = 63   *A1*

[1 mark]
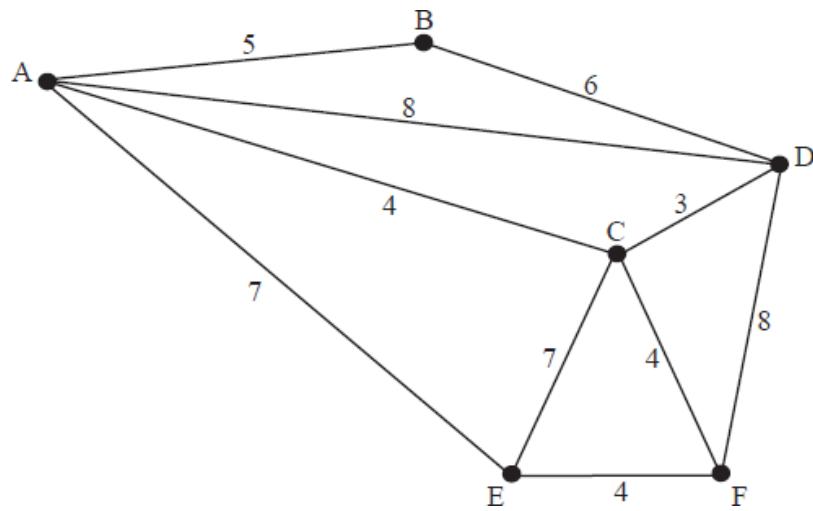
c. the conclusion is that ADCBEA gives a solution to the travelling salesman problem   *A1*

[1 mark]

Examiners report

a. This question was generally well answered although some candidates failed to realise the significance of the equality of the upper and lower

bounds.

b. This question was generally well answered although some candidates failed to realise the significance of the equality of the upper and lower

bounds.

c. This question was generally well answered although some candidates failed to realise the significance of the equality of the upper and lower

bounds.

Sameer is trying to design a road system to connect six towns, A, B, C, D, E and F. The possible roads and the costs of building them are shown in

the graph below. Each vertex represents a town, each edge represents a road and the weight of each edge is the cost of building that road. He needs

to design the lowest cost road system that will connect the six towns.

(a)   Name an algorithm which will allow Sameer to find the lowest cost road system.

(b)   Find the lowest cost road system and state the cost of building it. Show clearly the steps of the algorithm.

Markscheme

(a)   **EITHER**

Prim's algorithm   *A1*

OR

Kruskal's algorithm   *A1*

[1 mark]

(b)   **EITHER**

using Prim's algorithm, starting at A

Edge	Cost	
AC	4	*A1*
CD	3	*A1*
CF	4	*A1*
FE	4	*A1*
AB	5	*A1*

lowest cost road system contains roads AC, CD, CF, FE and AB   *A1*

cost is 20   *A1*

OR

using Kruskal's algorithm

Edge	Cost	
CD	3	*A1*
CF	4	*A1*
FE	4	*A1*
AC	4	*A1*
AB	5	*A1*

lowest cost road system contains roads CD, CF, FE, AC and AB   *A1*

cost is 20   *A1*

Note: Accept alternative correct solutions.

[7 marks]

Total [8 marks]

Examiners report

Most candidates were able to name an algorithm to find the lowest cost road system and then were able apply the algorithm. All but the weakest candidates were able to make a meaningful start to this question. In 1(b) some candidates lost marks by failing to indicate the order in which edges were added.

Consider the complete bipartite graph $\kappa_{3,\,3}$.

a.i. Draw $\kappa_{3,\,3}$. [1]

a.ii. Show that $\kappa_{3,\,3}$ has a Hamiltonian cycle. [1]

a.iii Draw $\kappa_{3,\,2}$ and explain why it does not have a Hamiltonian cycle. [2]
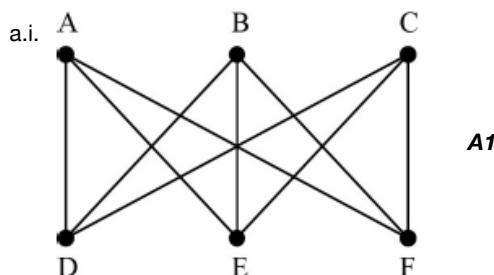
b.i. In the context of graph theory, state the handshaking lemma. [1]

b.ii. Hence show that a graph G with degree sequence 2, 3, 3, 4, 4, 5 cannot exist. [2]

c. Let T be a tree with v where $v \geq 2$. [4]

Use the handshaking lemma to prove that T has at least two vertices of degree one.

Markscheme

a.i.

A1
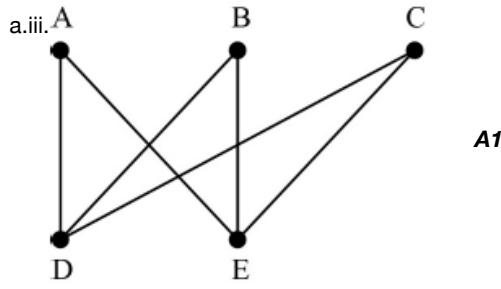
[1 mark]

a.ii. for example ADBECFA   *A1*

Note: Accept drawing the cycle on their diagram.

Note: Accept Dirac's theorem (although it is not on the syllabus for (a)(ii). There is no converse that could be applied for (a)(iii).

[1 mark]

a.iii.

A1

a Hamiltonian cycle would have to alternate between the two vertex subsets which is impossible as 2 ≠ 3   **R1**

Note: Award **R1** for an attempt to construct a Hamiltonian cycle and an explanation of why it fails, *eg*, ADBEC but there is no route from C to A without re-using D or E so no cycle. There are other proofs *eg*, have to go in and out of A, similarly B and C giving all edges leading to a contradiction.

[2 marks]

b.i. the sum of the vertex degrees is twice the number of edges   **A1**

[1 mark]

b.ii. assume G exists

the sum $2 + 3 + 3 + 4 + 4 + 5 = 21$ **A1**

this is odd (not even)   **R1**

this contradicts the handshaking lemma

so G does not exist   **AG**

[2 marks]

c. T has $v - 1$ edges   **A1**

EITHER

if k vertices have degree 1 then $v - k$ vertices have degree ≥ 2 **R1**

by the handshaking lemma

$2v - 2 \geqslant 1 \times k + 2\,(v - k)\,(= 2v - k)$ **M1**

this gives $k \geq 2$ **A1**

OR

let S be the sum of vertex degrees

consider T having either no or one vertex of degree 1   **R1**

case 1 suppose T has no vertices of degree 1 (*eg*, all vertices have degrees ≥ 2)

by the handshaking lemma

$S \geqslant 2v \neq 2\,(v - 1)$ (not possible)   **A1**

case 2 suppose T has one vertex of degree 1 (*eg*, $v - 1$ vertices have degrees ≥ 2)

by the handshaking lemma

$S \geqslant 2\,(v - 1) + 1 \neq 2\,(v - 1)$ (not possible)   **A1**

THEN

so T has at least two vertices of degree 1   **AG**
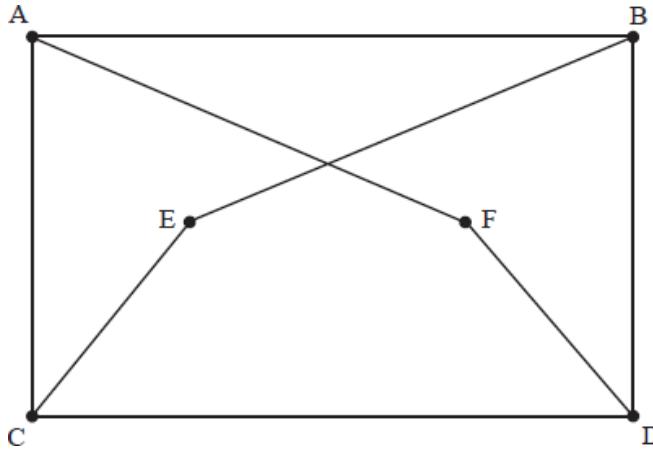
[4 marks]

Examiners report

a. Draw the complement of the following graph as a planar graph. [3]
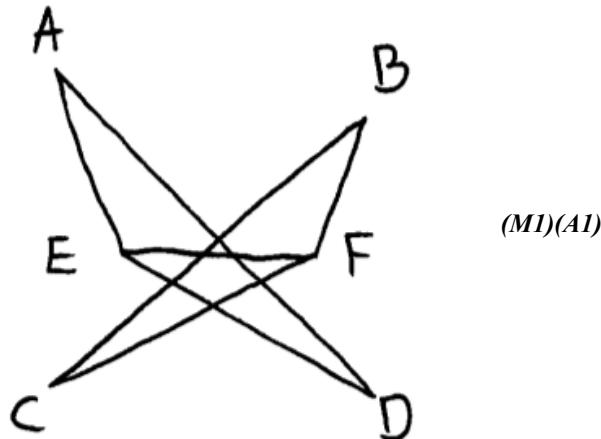
b. A simple graph G has v vertices and e edges. The complement G' of G has e' edges. [14]
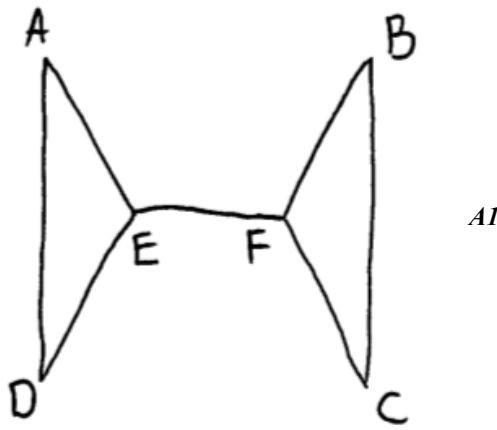
   (i)   Prove that $e \leqslant \frac{1}{2}v(v-1)$.

   (ii)   Find an expression for $e + e'$ in terms of v .

   (iii)   Given that G' is isomorphic to G , prove that v is of the form $4n$ or $4n + 1$ for $n \in \mathbb{Z}^+$.

   (iv)   Prove that there is a unique simple graph with 4 vertices which is isomorphic to its complement.

   (v)   Prove that if $v \geqslant 11$, then G and G' cannot both be planar.

Markscheme

a. as a first step, form the following graph

      (M1)(A1)

make it planar

A1

[3 marks]

b. (i) an edge joins a pair of vertices **R1**

there is a maximum of $\begin{pmatrix} v \\ 2 \end{pmatrix} = \frac{1}{2}v(v-1)$ possible unordered pairs of vertices, hence displayed result **A1AG**

(ii) an edge joins two vertices in G' if it does not join them in G and *vice versa*; all possible edges are accounted for by the union of the two graphs **R1**
$e + e' = \frac{1}{2}v(v-1)$ **A1**

(iii) the two graphs have the same number of edges **R1**
$\Rightarrow e = \frac{1}{4}v(v-1)$ **A1**
v and $v-1$ are consecutive integers, so only one can be divisible by 4, hence displayed result **A1AG**

(iv) the required graphs have four vertices and three edges **A1**
if one vertex is adjacent to the other three, that uses up the edges; the resulting graph, necessarily connected, has a disconnected complement, and *vice versa* **R1**
if one vertex is adjacent to two others, that uses up two edges; the final vertex cannot be adjacent to the first; the result is the linear connected graph **A1**
state it is isomorphic to its complement **A1** **N2**
Note: Alternative proofs are possible, but should include the final statement for full marks.

(v) using $e \leqslant 3v - 6$ for planar graphs **M1**
$\frac{1}{2}v(v-1) = e + e' \leqslant 6v - 12$ **A1**
$v^2 - 13v + 24 \leqslant 0$ is not possible for $v \geqslant 11$ **R1**
[14 marks]

Examiners report

a. Part (a) was well done. The various parts of Parts (b) were often attempted, but with a disappointing feeling that the candidates did not have a confident understanding of what they were writing.

b. Part (a) was well done. The various parts of Parts (b) were often attempted, but with a disappointing feeling that the candidates did not have a confident understanding of what they were writing.

(a) A connected planar graph G has e edges and v vertices.

(i) Prove that $e \geqslant v - 1$.

(ii) Prove that $e = v - 1$ if and only if G is a tree.

(b) A tree has k vertices of degree 1, two of degree 2, one of degree 3 and one of degree 4. Determine k and hence draw a tree that satisfies these conditions.

(c)   The graph H has the adjacency table given below.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(i)   Explain why H cannot be a tree.

(ii)   Draw the graph of H.

(d)   Prove that a tree is a bipartite graph.

Markscheme

(a)   (i)   Euler's relation is

$e = v - 2 + f \geqslant v - 1$, as $f \geqslant 1$ *M1A1*

(ii) G is a tree $\Leftrightarrow$ no cycles $\Leftrightarrow f = 1$ *R1R1*

[4 marks]

(b)   the result from (a) (ii) gives

$e = k + 2 + 1 + 1 - 1 = k + 3$ *M1A1*
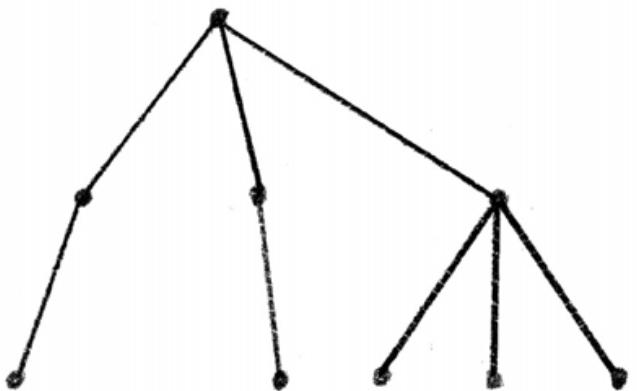
for a tree we also have

$2e =$ sum of degrees   *M1*

$2k + 6 = k + 4 + 3 + 4 = k + 11$

hence $k = 5$ *A1*

A2
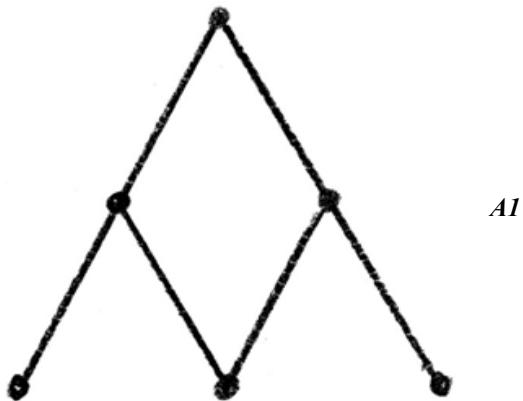
Note: Accept alternative correct solutions.

[6 marks]

(c) (i) $v - 1 = 5 < 6 = e$ by (a) (ii)   *M1A1*

G cannot be a tree   *AG*

(ii)   *A1*

[3 marks]

(d)   take any vertex in the tree and colour it black   *M1*

colour all adjacent vertices white

colour all vertices adjacent to a white vertex black

continue this procedure until all vertices are coloured   *M1*

which must happen since the graph is connected   *R1*

as the tree contains no cycles, no vertex can be both black and white and the graph is proved to be bipartite   *R1*
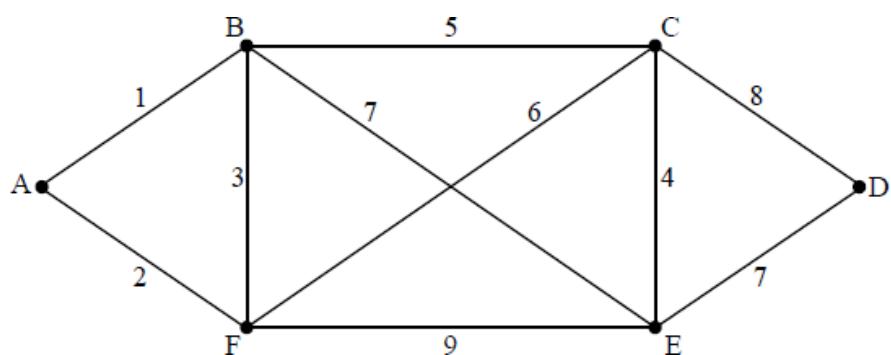
[4 marks]

Total [17 marks]

Examiners report

Many candidates seem only to have a weak understanding of the requirements for the proof of a mathematical statement.

The diagram below shows the weighted graph G .
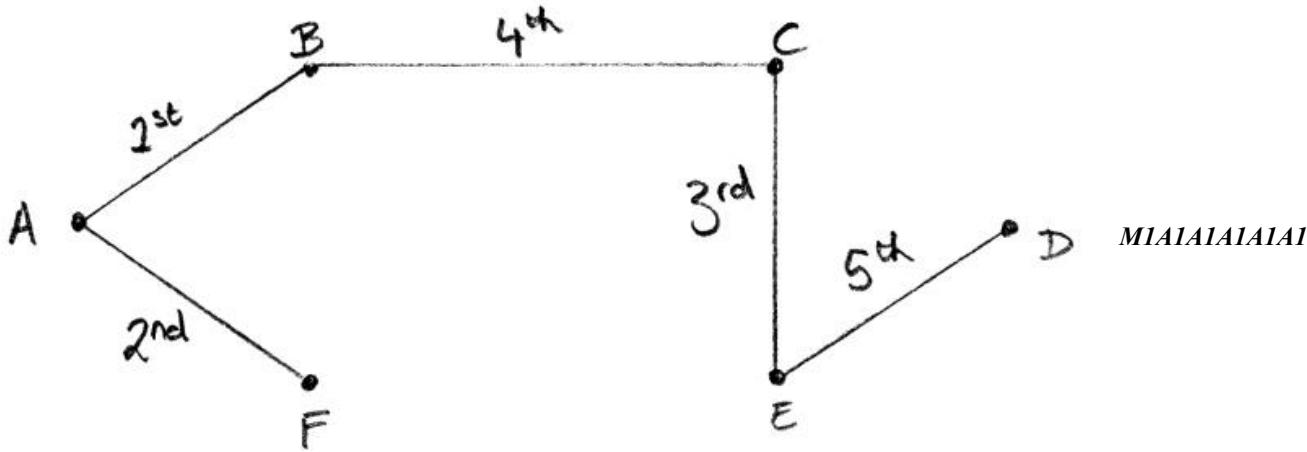
(a) (i) What feature of the graph enables you to deduce that G contains an Eulerian circuit?

(ii) Find an Eulerian circuit.

(c) Use Kruskal's Algorithm to find the minimum spanning tree for G, showing the order in which the edges are added.

Markscheme

(a) (i) all the vertices have even degree *A1*

(ii) for example ABCDECFBEFA *A2*

[3 marks]

(c) the edges are included in the order shown

Note: Award each *A1* for the edge added in the correct order. Award no further marks after the first error.

[6 marks]

Total [9 marks]

Examiners report

Part (a) was well answered in general. Part (c) was well answered.
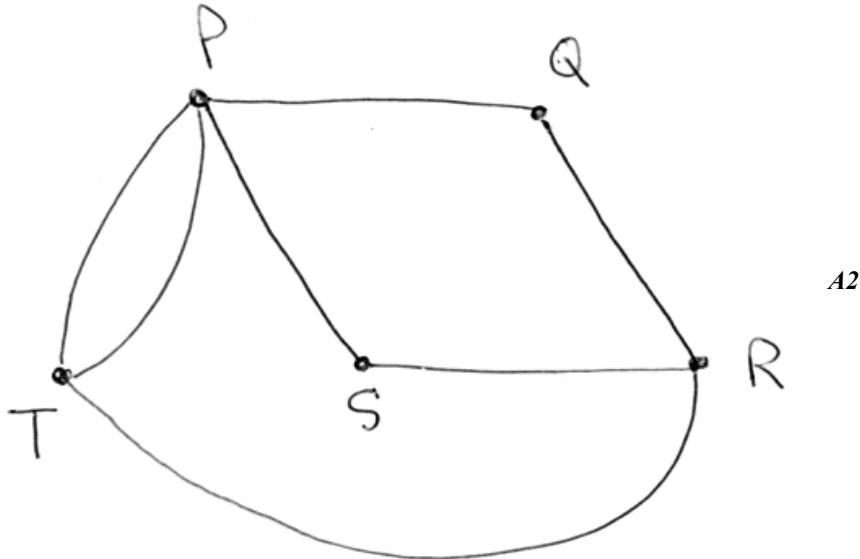
The graph G has vertices P, Q, R, S, T and the following table shows the number of edges joining each pair of vertices.

	P	Q	R	S	T
P	0	1	0	1	2
Q	1	0	1	0	0
R	0	1	0	1	1
S	1	0	1	0	0
T	2	0	1	0	0

a. Draw the graph *G* as a planar graph. [2]

b. Giving a reason, state whether or not *G* is [4]

    (i)   simple;

    (ii)   connected;

    (iii)   bipartite.

c. Explain what feature of *G* enables you to state that it has an Eulerian trail and write down a trail. [2]

d. Explain what feature of *G* enables you to state it does not have an Eulerian circuit. [1]

e. Find the maximum number of edges that can be added to the graph *G* (not including any loops or further multiple edges) whilst still keeping [4] it planar.

Markscheme

a.

A2

[2 marks]

b. (i)   *G* is not simple because 2 edges join P to T   *R1*

    (ii)   *G* is connected because there is a path joining every pair of vertices   *R1*

    (iii)   (P, R) and (Q, S, T) are disjoint vertices   *R1*
    so *G* is bipartite   *A1*
    Note: Award the *A1* only if the *R1* is awarded.

[4 marks]

c. *G* has an Eulerian trail because it has two vertices of odd degree (R and T have degree 3), all the other vertices having even degree   *R1*

the following example is such a trail
TPTRSPQR   *A1*
[2 marks]
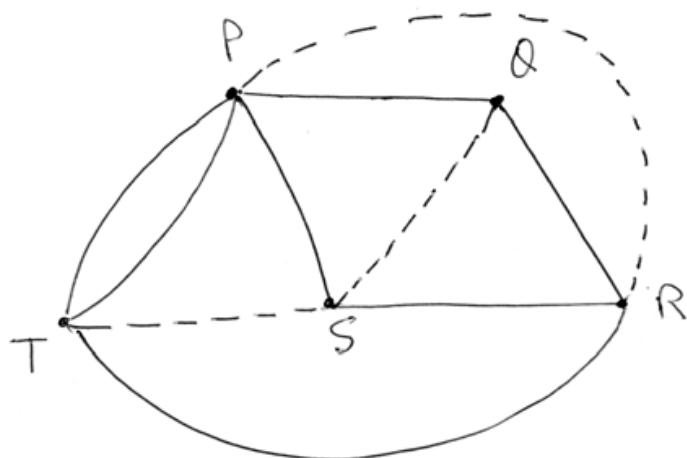
d. *G* has no Eulerian circuit because there are 2 vertices which have odd degree   *R1*

[1 mark]

e. consider

so it is possible to add 3 extra edges   *A1*

consider G with one of the edges PT deleted; this is a simple graph with 6 edges; on addition of the new edges, it will still be simple   *M1*

$e \leqslant 3v - 6 \Rightarrow e \leqslant 3 \times 5 - 6 = 9$ *R1*

so at most 3 edges can be added   *R1*

[4 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]

(i)   A graph is simple, planar and connected. Write down the inequality connecting v and e, and give the condition on v for this inequality to hold.

(ii)   Sketch a simple, connected, planar graph with $v = 2$ where the inequality from part (b)(i) is not true.

(iii)   Sketch a simple, connected, planar graph with $v = 1$ where the inequality from part (b)(i) is not true.

(iv)   Given a connected, planar graph with v vertices, v^2 edges and 8 faces, find v. Sketch a graph that fulfils all of these conditions.

Markscheme

(i) $e \leqslant 3v - 6$, for $v \geqslant 3$ *A1A1*

(ii)   *A1*

(iii)   *A1*

(iv)   from Euler's relation $v - e + f = 2$
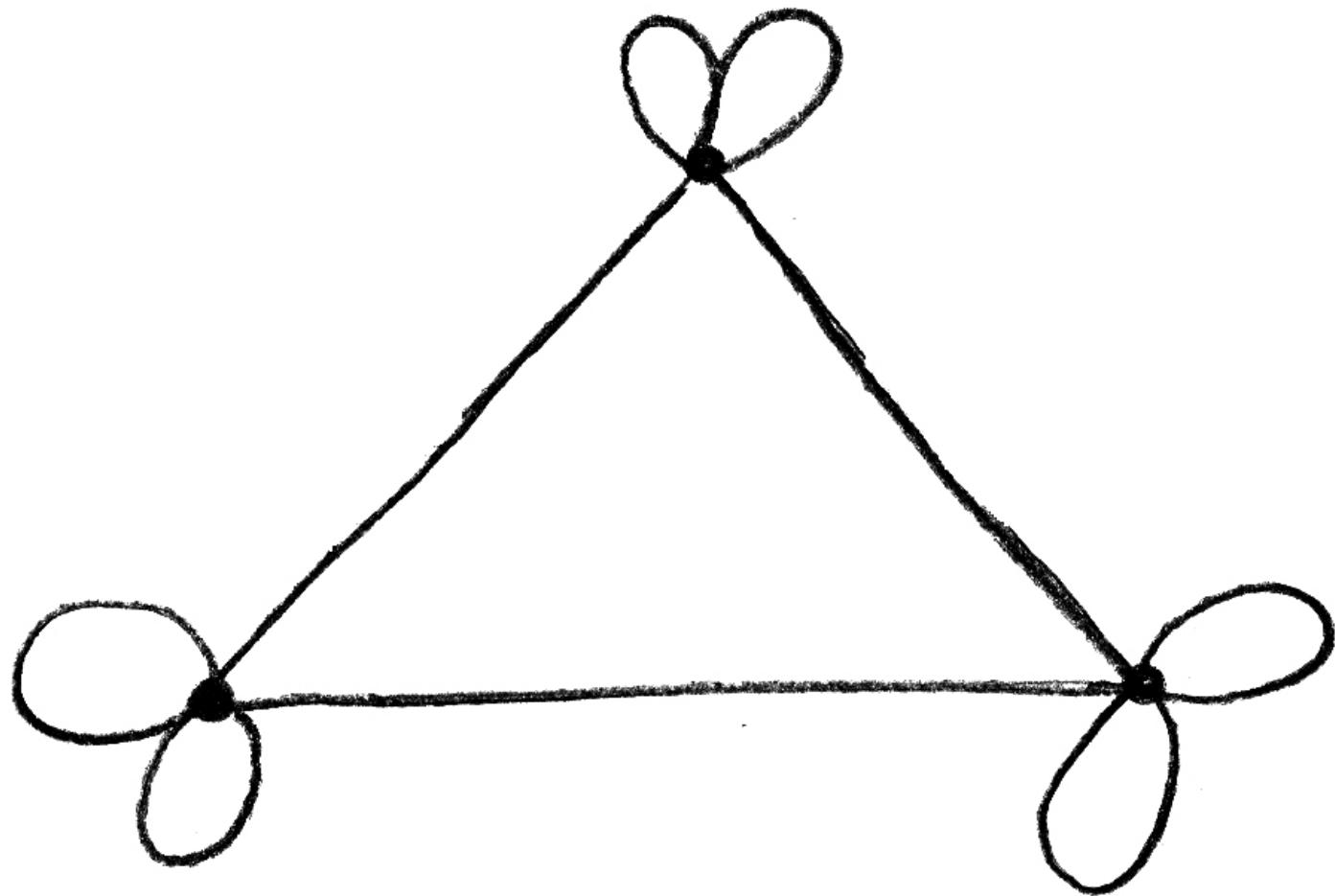
$v - v^2 + 8 = 2$ *M1*

$v^2 - v - 6 = 0$ *A1*

$(v+2)(v-3) = 0$

$v = 3$ *A1*

for example

A1

Note: There are many possible graphs.

[8 marks]

Examiners report

In (b) most candidates gave the required inequality although some just wrote down both inequalities from their formula booklet. The condition $v \geqslant 3$ was less well known but could be deduced from the next 2 graphs. Euler's relation was used well to obtain the quadratic to solve and many candidates could then draw a correct graph.

---

a. The table below shows the distances between towns A, B, C, D and E.   [9]
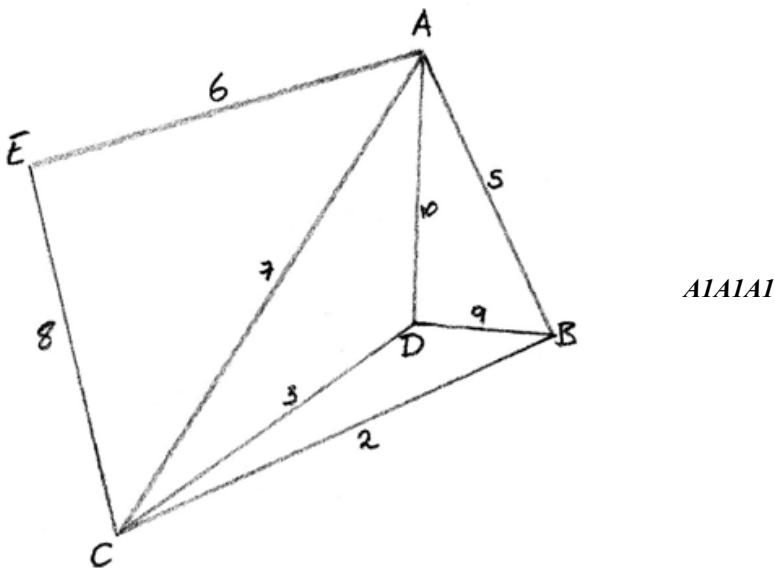
$$\begin{array}{c c c c c c}
 & A & B & C & D & E \\
A & - & 5 & 7 & 10 & 6 \\
B & 5 & - & 2 & 9 & - \\
C & 7 & 2 & - & 3 & 8 \\
D & 10 & 9 & 3 & - & - \\
E & 6 & - & 8 & - & -
\end{array}$$

(i) Draw the graph, in its planar form, that is represented by the table.

(ii) Write down with reasons whether or not it is possible to find an Eulerian trail in this graph.

(iii) Solve the Chinese postman problem with reference to this graph if A is to be the starting and finishing point. Write down the walk and determine the length of the walk.

b. Show that a graph cannot have exactly one vertex of odd degree. [2]

Markscheme

a. (i)
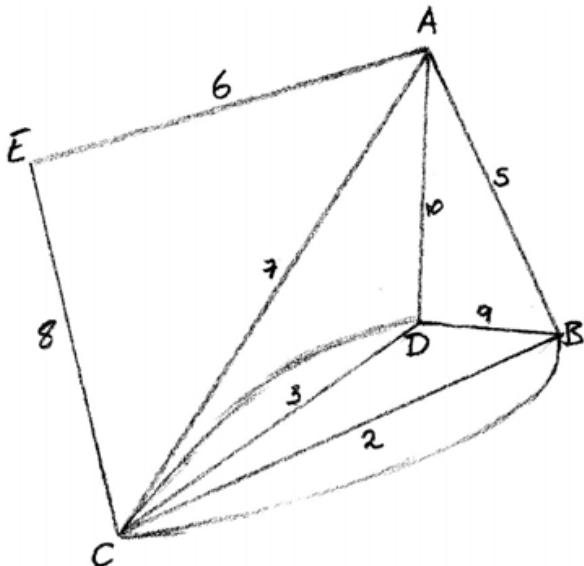
Note: Award **A1** for the vertices, **A1** for edges and **A1** for planar form.

(ii) It is possible to find an Eulerian trail in this graph since exactly two of the vertices have odd degree **R1**

(iii) B and D are the odd vertices **M1**

$BC + CD = 3 + 2 = 5$ and $BD = 9$, **A1**

since $5 < 9$, BC and CD must be traversed twice **R1**

A possible walk by inspection is ACBDABCDCEA   *A1*

This gives a total length of

$2(2 + 3) + 8 + 9 + 5 + 7 + 10 + 6 = 55$ for the walk   *A1*
[9 marks]

b. The sum of all the vertex degrees is twice the number of edges, *i.e.* an even number.

Hence a graph cannot have exactly one vertex of odd degree.   *M1R1*
[2 marks]

Examiners report

a. Drawing the graph usually presented no difficulty. Distinguishing between Eulerian and semi-Eulerian needs attention but this part was usually

done successfully.

A simple, clear argument for part (c) was often hidden in mini-essays on graph theory.

b. Drawing the graph usually presented no difficulty. Distinguishing between Eulerian and semi-Eulerian needs attention but this part was usually

done successfully.

A simple, clear argument for part (c) was often hidden in mini-essays on graph theory.

A graph G with vertices A, B, C, D, E has the following cost adjacency table.

	A	B	C	D	E
A	–	12	10	17	19
B	12	–	13	20	11
C	10	13	–	16	14
D	17	20	16	–	15
E	19	11	14	15	–

(a) (i)   Use Kruskal's algorithm to find and draw the minimum spanning tree for G.

(ii)   The graph H is formed from G by removing the vertex D and all the edges connected to D. Draw the minimum spanning tree for H and use it to find a lower bound for the travelling salesman problem for G.

(b)   Show that 80 is an upper bound for this travelling salesman problem.

# Markscheme
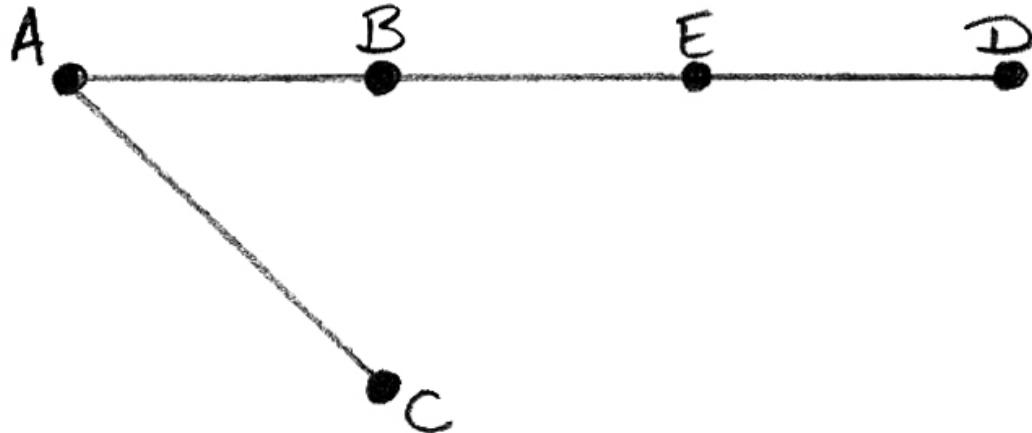
(a) (i) the edges are joined in the order

AC

BE

AB

ED   *A2*

A1
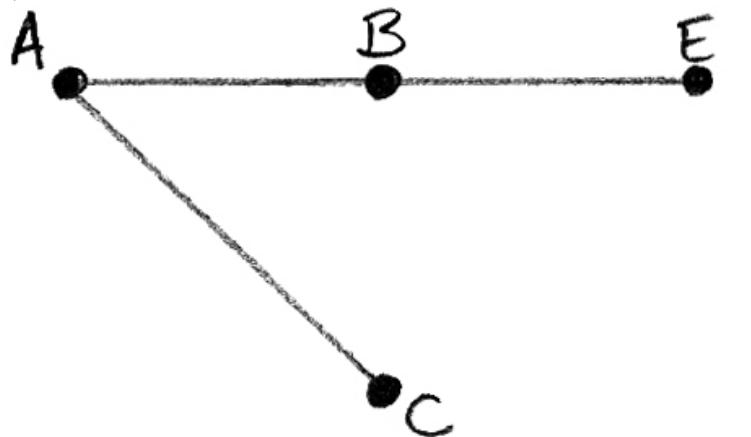
Note: Final *A1* independent of the previous *A2*.

(ii)

A1

the weight of this spanning tree is 33   *A1*

to find a lower bound for the travelling salesman problem, we add to that the two smallest weights of edges to D, *i.e.* 15 +16, giving 64   *M1A1*

[7 marks]

(b) an upper bound is the weight of any Hamiltonian cycle, *e.g.* ABCDEA has weight 75 so 80 is certainly an upper bound   *M1A1*
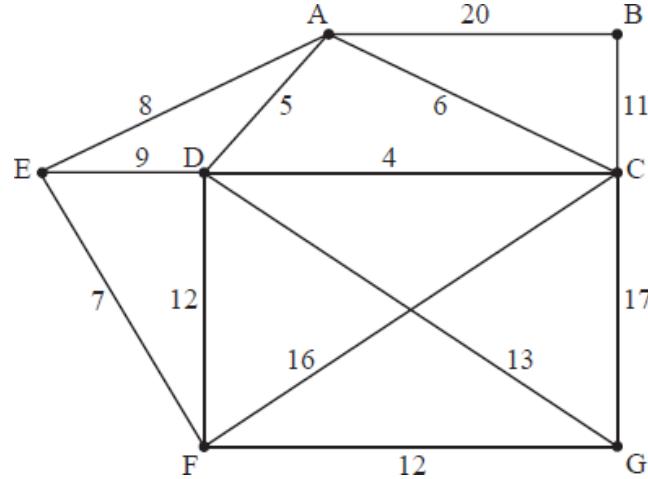
[2 marks]

Total [9 marks]

Examiners report
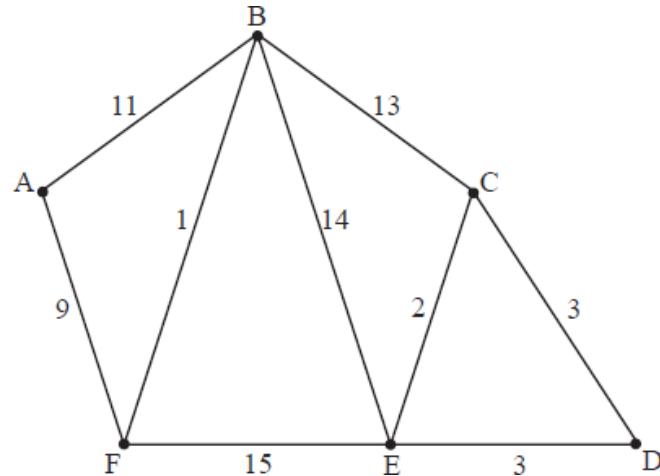
Part (a) was well done by many candidates although some candidates simply drew the minimum spanning tree in (i) without indicating the use of

Kruskal's Algorithm. It is important to stress to candidates that, as indicated in the rubric at the top of Page 2, answers must be supported by

working and/or explanations. Part (b) caused problems for some candidates who obtained the unhelpful upper bound of 96 by doubling the weight

of the minimum spanning tree. It is useful to note that the weight of any Hamiltonian cycle is an upper bound and in this case it was fairly easy to find such a cycle with weight less than or equal to 80.

a. Use Kruskal's algorithm to find the minimum spanning tree for the following weighted graph and state its length. [5]


b. Use Dijkstra's algorithm to find the shortest path from A to D in the following weighted graph and state its length. [7]

Markscheme

a. Kruskal's algorithm gives the following edges

CD (4) *M1A1*

AD (5)

EF (7) *A1*

EA (8)

BC (11)

FG (12) *A1* *N0*
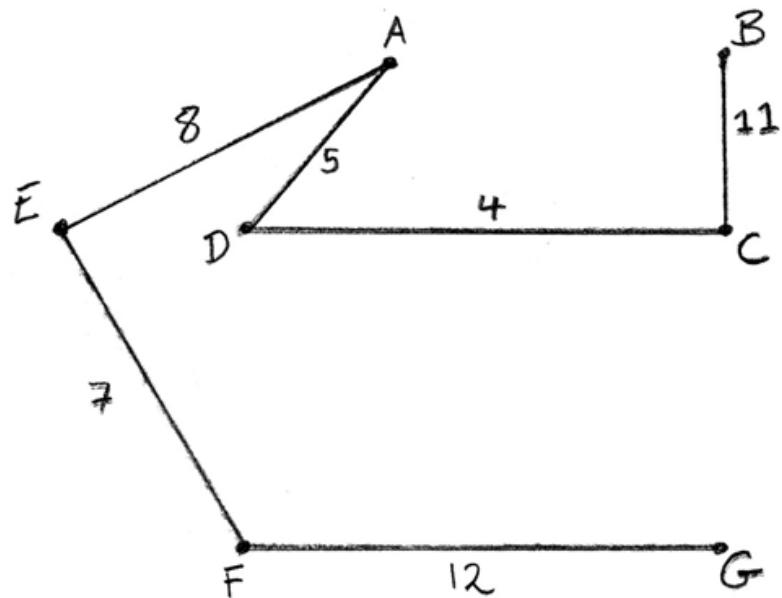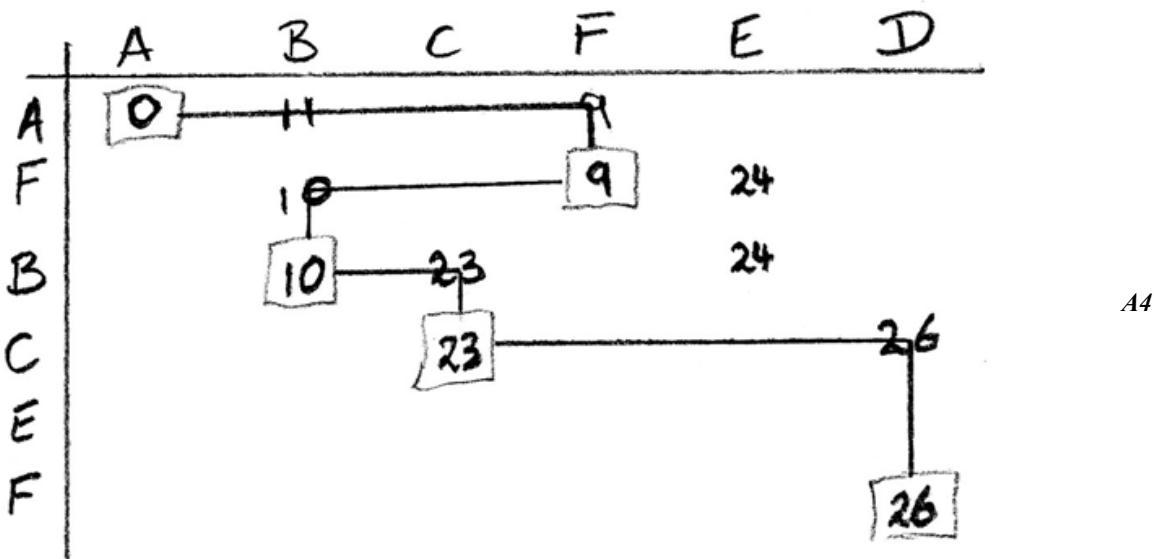
length of the spanning tree is 47   *A1*

[5 marks]

b. for Dijkstra's algorithm there are three things associated with a node: order; distance from the initial node as a permanent or temporary node

M1

A4

Note: Deduct *A1* for each error or omission.

the shortest path is AFBCD   *A1*

the length is 26   *A1*   *N0*

[7 marks]

Examiners report

a. Setting out clearly the steps of the algorithms is still a problem for many although getting the correct spanning tree and its length were not.

b. Setting out clearly the steps of the algorithms is still a problem for many although getting the correct spanning tree and its length were not.
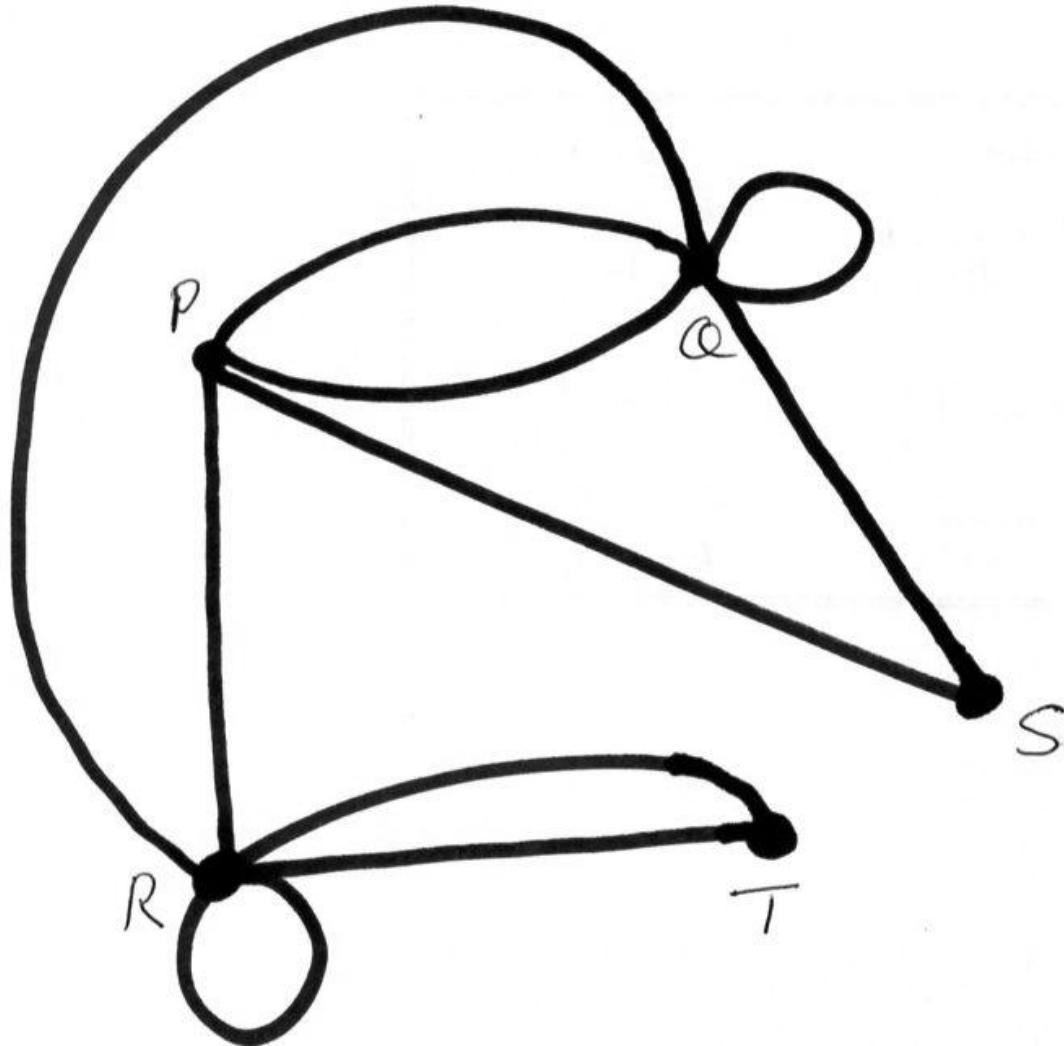
The adjacency table of the graph G, with vertices P, Q, R, S, T is given by:

$$
\begin{array}{c c c c c c}
 & P & Q & R & S & T \\
P & 0 & 2 & 1 & 1 & 0 \\
Q & 2 & 1 & 1 & 1 & 0 \\
R & 1 & 1 & 1 & 0 & 2 \\
S & 1 & 1 & 0 & 0 & 0 \\
T & 0 & 0 & 2 & 0 & 0
\end{array}
$$

(a)   Draw the graph G.

(b)   (i)   Define an Eulerian circuit.

    (ii)   Write down an Eulerian circuit in G starting at P.

(c)   (i)   Define a Hamiltonian cycle.

    (ii)   Explain why it is not possible to have a Hamiltonian cycle in G.

Markscheme

(a)

A3

Note: Award *A2* for one missing or misplaced edge,

A1 for two missing or misplaced edges.

[3 marks]

(b) (i) an Eulerian circuit is one that contains every edge of the graph exactly once   *A1*

(ii) a possible Eulerian circuit is

$P \rightarrow Q \rightarrow S \rightarrow P \rightarrow Q \rightarrow Q \rightarrow R \rightarrow T \rightarrow R \rightarrow R \rightarrow P$ *A2*

[3 marks]

(c) (i) a Hamiltonian cycle passes through each vertex of the graph   *A1*

exactly once   *A1*
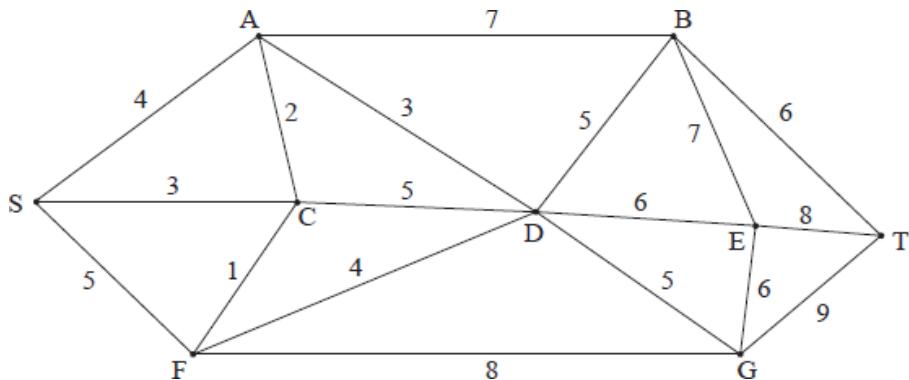

(ii) to pass through T, you must have come from R and must return to R.   *R3*

hence there is no Hamiltonian cycle

[5 marks]

Examiners report

Stronger candidates had little problem with this question, but a significant number of weaker candidates started by making errors in drawing the

graph G, where the most common error was the omission of the loops and double edges. They also had problems working with the concepts of

Eulerian circuits and Hamiltonian cycles.

In the graph given above, the numbers shown represent the distance along that edge.

a. Using Dijkstra's algorithm, find the length of the shortest path from vertex S to vertex T. Write down this shortest path.   [6]

b. (i)   Does this graph have an Eulerian circuit? Justify your answer.   [4]

   (ii)   Does this graph have an Eulerian trail? Justify your answer.
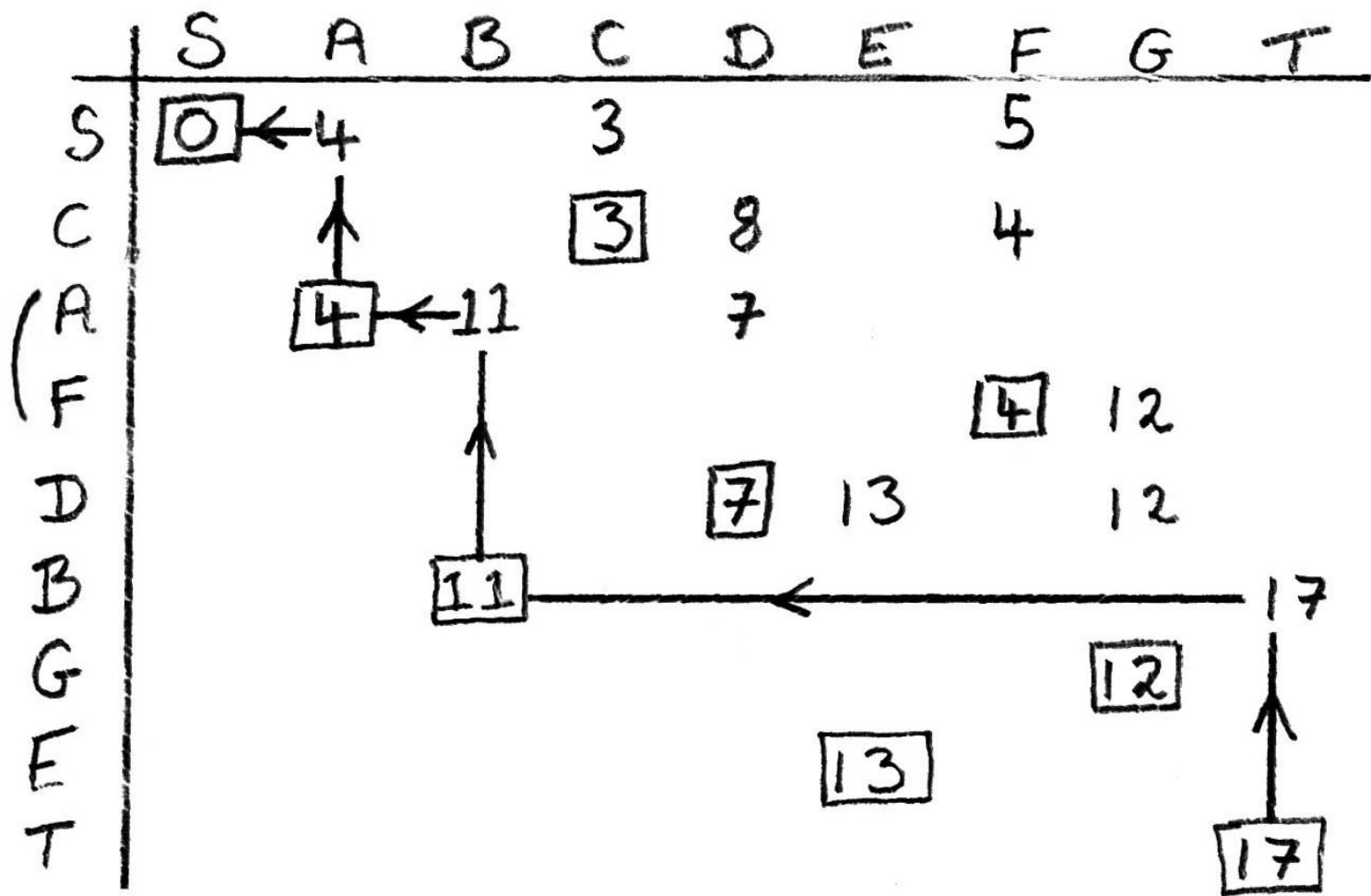
c. The graph above is now to be considered with the edges representing roads in a town and with the distances being the length of that road in   [4]

kilometres. Huan is a postman and he has to travel along every road in the town to deliver letters to all the houses in that road. He has to

start at the sorting office at S and also finish his route at S. Find the shortest total distance of such a route. Fully explain the reasoning

behind your answer.

Markscheme

a.

from tabular method as shown above (or equivalent)   *M1A1A1*
Note: Award the first *A1* for obtaining 3 as the shortest distance to C.
Award the second *A1* for obtaining the rest of the shortest distances.

shortest path has length 17   *A1*
backtracking as shown above (or equivalent)   *(M1)*
shortest path is SABT   *A1*
[6 marks]

b. (i)   no, as *S* and *T* have odd degree   *A1R1*

Note: Mentioning one vertex of odd degree is sufficient.

(ii)   yes, as only *S* and *T* have odd degree   *A1R1*
Note: In each case only award the *A1* if the *R1* has been given.
Accept an actual trail in (b)(ii).

[4 marks]

c. Huan has to travel along all the edges via an open Eulerian trail of length   *R1*

$4 + 3 + 5 + 2 + 1 + 3 + 5 + 4 + 7 + 8 + 5 + 6 + 7 + 6 + 6 + 8 + 9 = 94$ *A1*
and then back to *S* from *T* along the shortest path found in (a) of length 17   *R1*
so shortest total distance is $94 + 17 = 111$ *A1*
[4 marks]
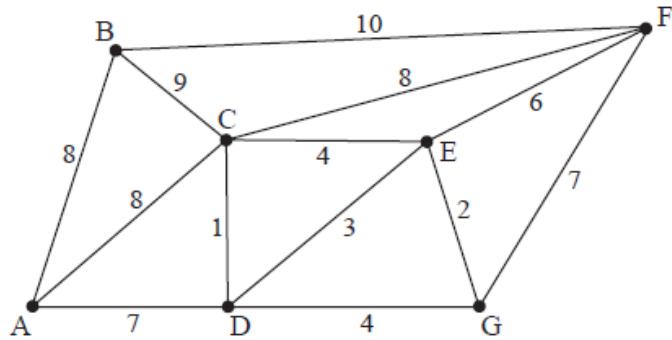
Examiners report

a. This was quite well answered. Some candidates did not make their method clear and others showed no method at all. Some clearly had a correct method but did not make it clear what their final answers were. It is recommended that teachers look at the tabular method with its backtracking system as shown in the mark scheme as an efficient way of tackling this type of problem.

b. Fairly good knowledge shown here but not by all.

c. Some good answers but too much confusion with methods they partly remembered about the travelling salesman problem. Candidates should be aware of helpful connections between parts of a question.

Consider the following weighted graph.

(a) (i) Use Kruskal's algorithm to find the minimum spanning tree. Indicate the order in which you select the edges and draw the final spanning tree.

(ii) Write down the total weight of this minimum spanning tree.

(b) Sketch a spanning tree of maximum total weight and write down its weight.

Markscheme

(a) (i) *(Kruskal's: successively take an edge of smallest weight without forming a cycle)*

1st edge DC (weight 1)   *A1*
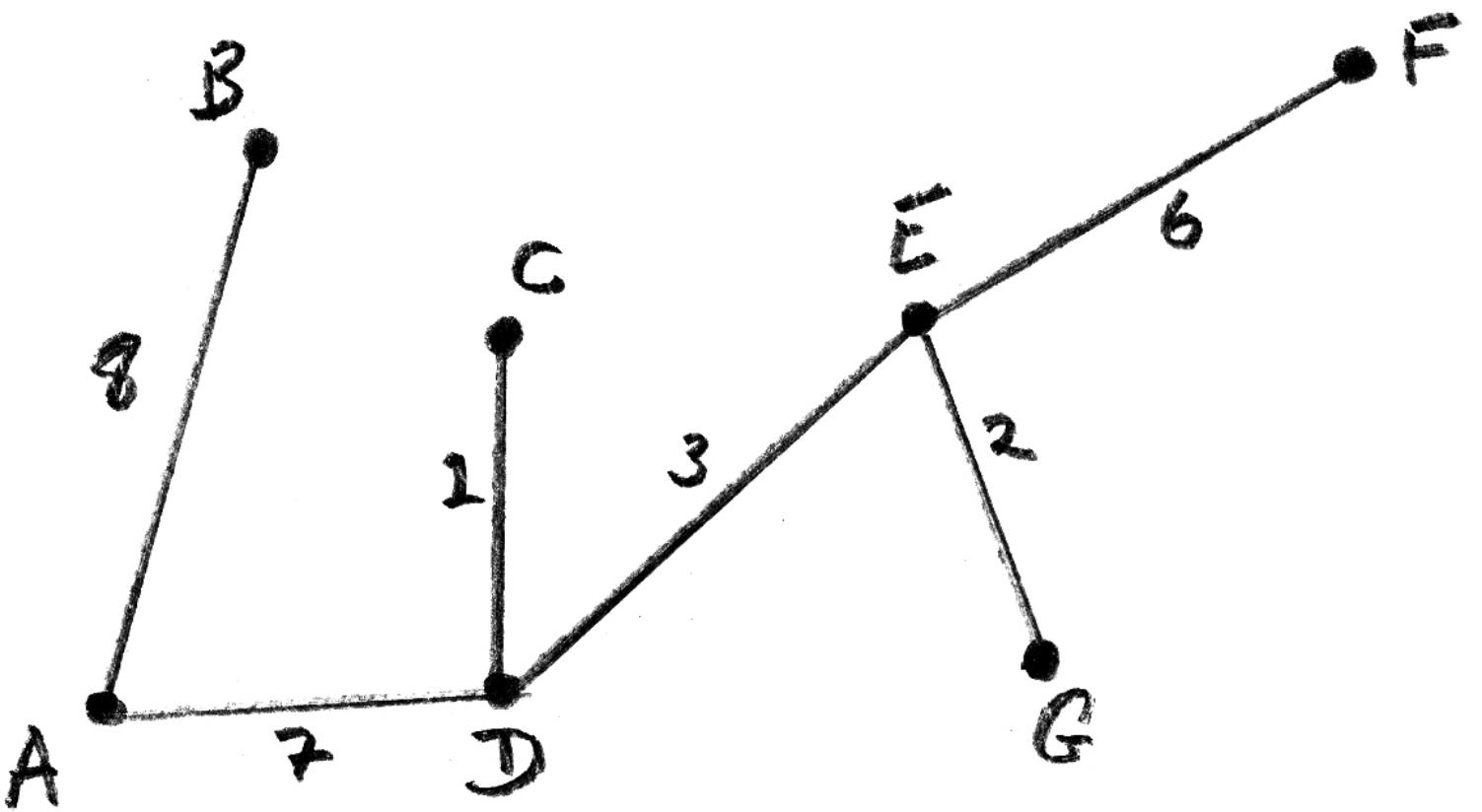
2nd edge EG (weight 2)   *A1*

3rd edge DE (weight 3)   *A1*

4th edge EF (weight 6)   *A1*

5th edge AD (weight 7)   *A1*

6th edge AB (weight 8)   *A1*

A1
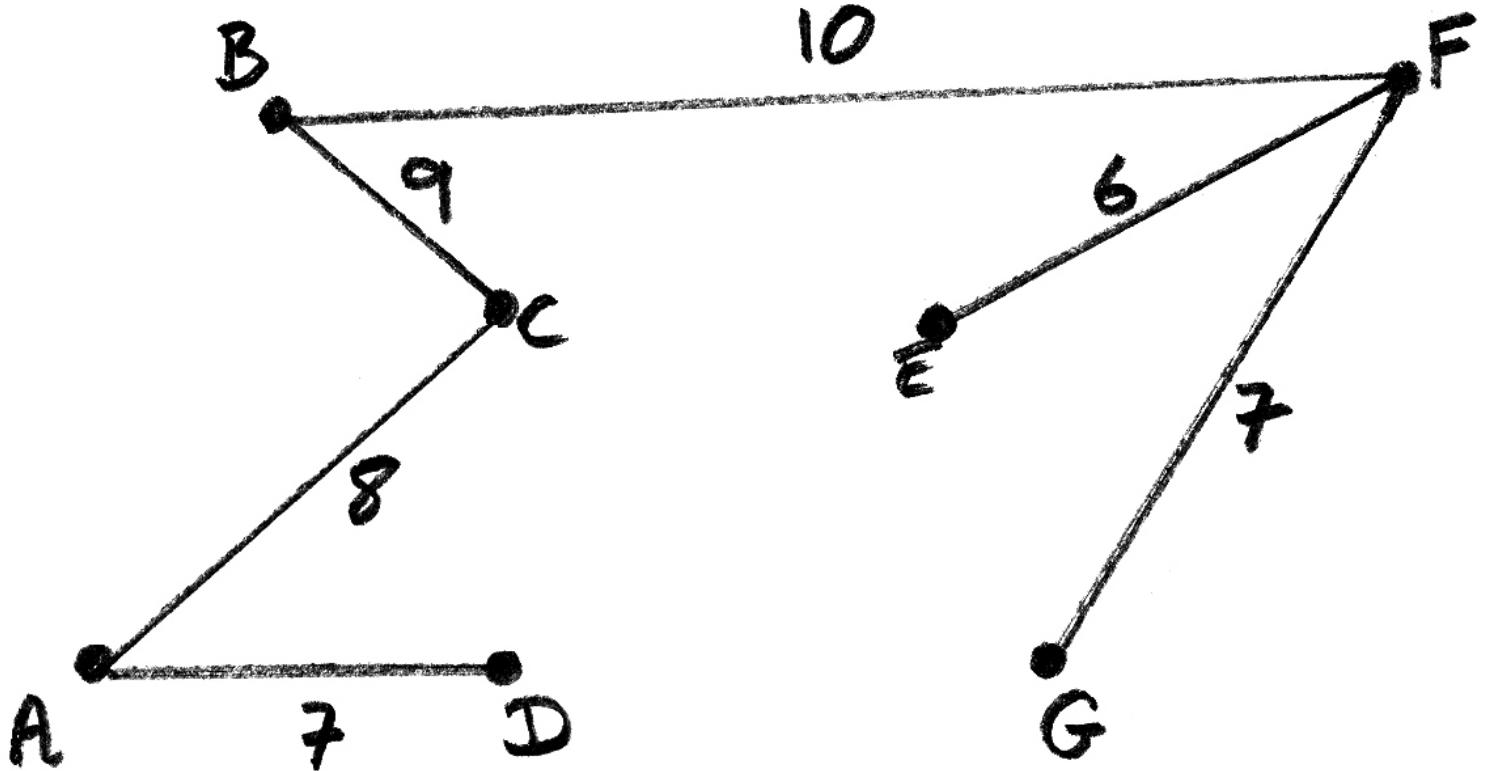
Notes: Weights are not required on the diagram.

Allow *A2(d)* if the (correct) edges are in the wrong order *e.g.* they have used Prim's rather than Kruskal's algorithm.

(ii)   total weight is $1 + 2 + 3 + 6 + 7 + 8 = 27$ *A1*
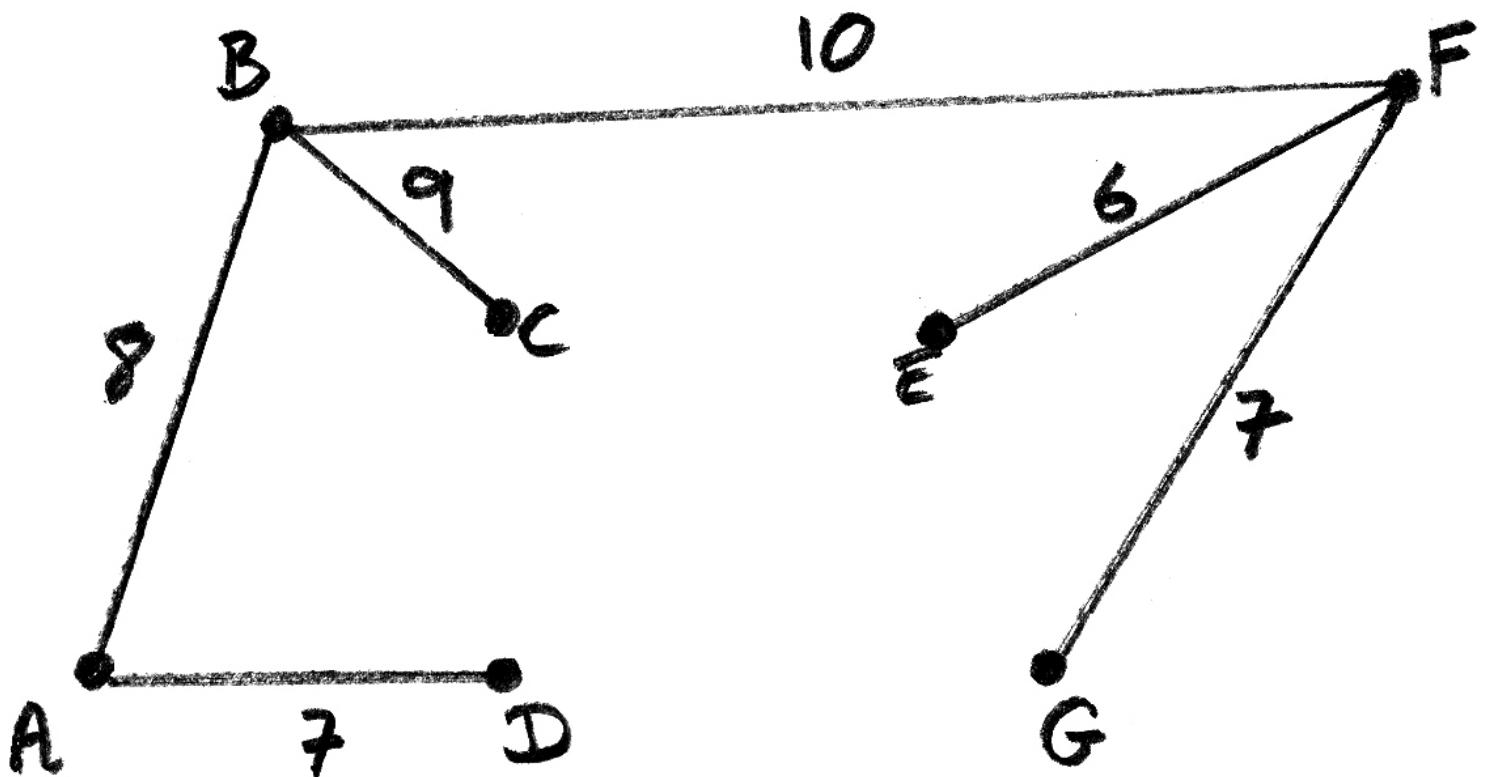
[8 marks]

(b)   **EITHER**

A3

OR

A3

Notes: Award *A2* for five or four correct edges,

A1 for three or two correct edges

A0 otherwise.

Weights are not required on the diagram.

THEN

total weight is $6 + 7 + 7 + 8 + 9 + 10 = 47$ *A1*

[4 marks]

Total [12 marks]

Examiners report
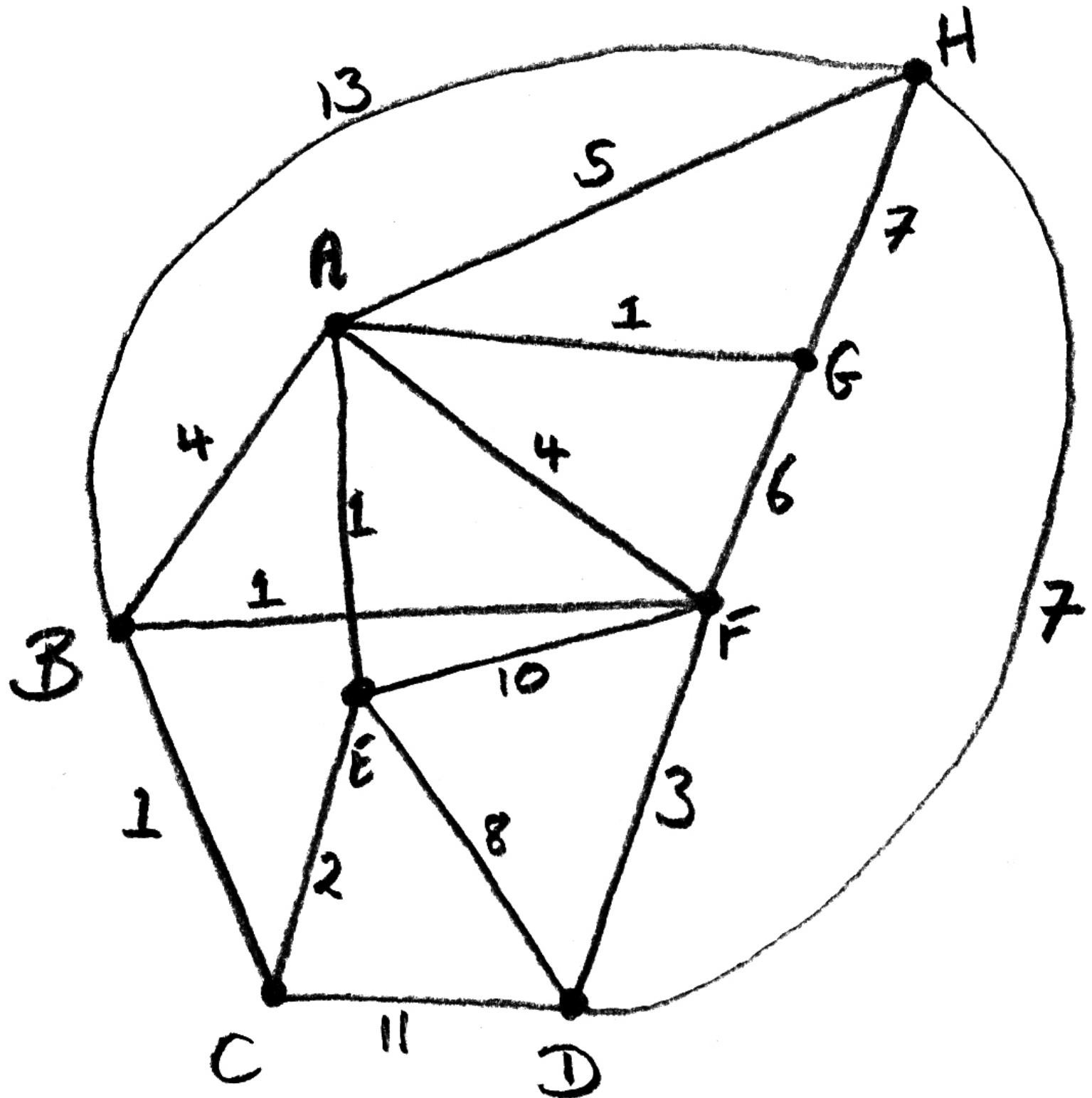
Good algorithm work was shown; sometimes there were mistakes in giving the order of the edges chosen by, for example doing Prim's algorithm

instead of Kruskal's.

The vertices of a graph L are A, B, C, D, E, F, G and H. The weights of the edges in L are given in the following table.

	A	B	C	D	E	F	G	H
A	–	4	–	–	1	4	1	5
B	4	–	1	–	–	1	–	13
C	–	1	–	11	2	–	–	–
D	–	–	11	–	8	3	–	7
E	1	–	2	8	–	10	–	–
F	4	1	–	3	10	–	6	–
G	1	–	–	–	–	6	–	7
H	5	13	–	7	–	–	7	–

Draw the graph L.

Markscheme

A2

Note: Award *A1* if one line missing or one line misplaced. Weights are not required.

[2 marks]

Examiners report

Most candidates were able to draw the graph as required in (a) and most made a meaningful start to applying Prim's algorithm in (b). Candidates were not always clear about the order in which the edges were to be added.

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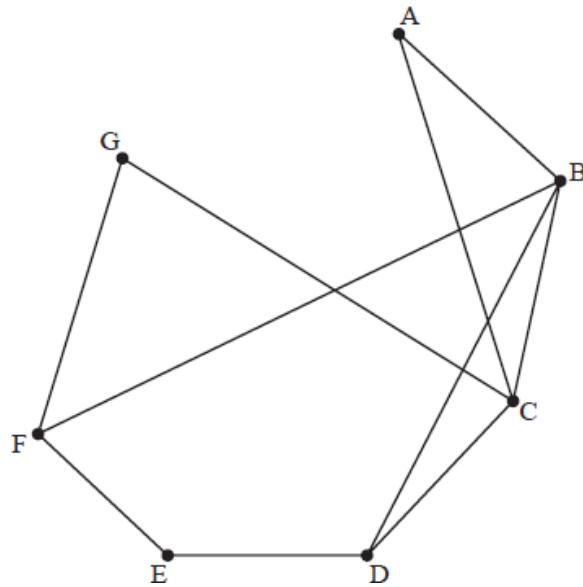
a. In any graph, show that [5]

   (i)   the sum of the degrees of all the vertices is even;

   (ii) there is an even number of vertices of odd degree.

b. Consider the following graph, M. [12]

   (i)   Show that M is planar.

   (ii)   Explain why M is not Eulerian.

   (iii) By adding one edge to M it is possible to make it Eulerian. State which edge must be added.

   This new graph is called N.

   (iv)   Starting at A, write down a possible Eulerian circuit for N.

   (v)   Define a Hamiltonian cycle. If possible, write down a Hamiltonian cycle for N, and if not possible, give a reason.

   (vi)   Write down the adjacency matrix for N.

   (vii) Which pair of distinct vertices has exactly 30 walks of length 4 between them?

Markscheme

a. (i)   When we sum over the degrees of all vertices, we count each edge twice. Hence every edge adds two to the sum. Hence the sum of the degrees of all the vertices is even.   *R2*

   (ii)   divide the vertices into two sets, those with even degree and those with odd degree   *M1*

   let S be the sum of the degrees of the first set and let T be the sum of the degrees of the second set

   we know $S + T$ must be even
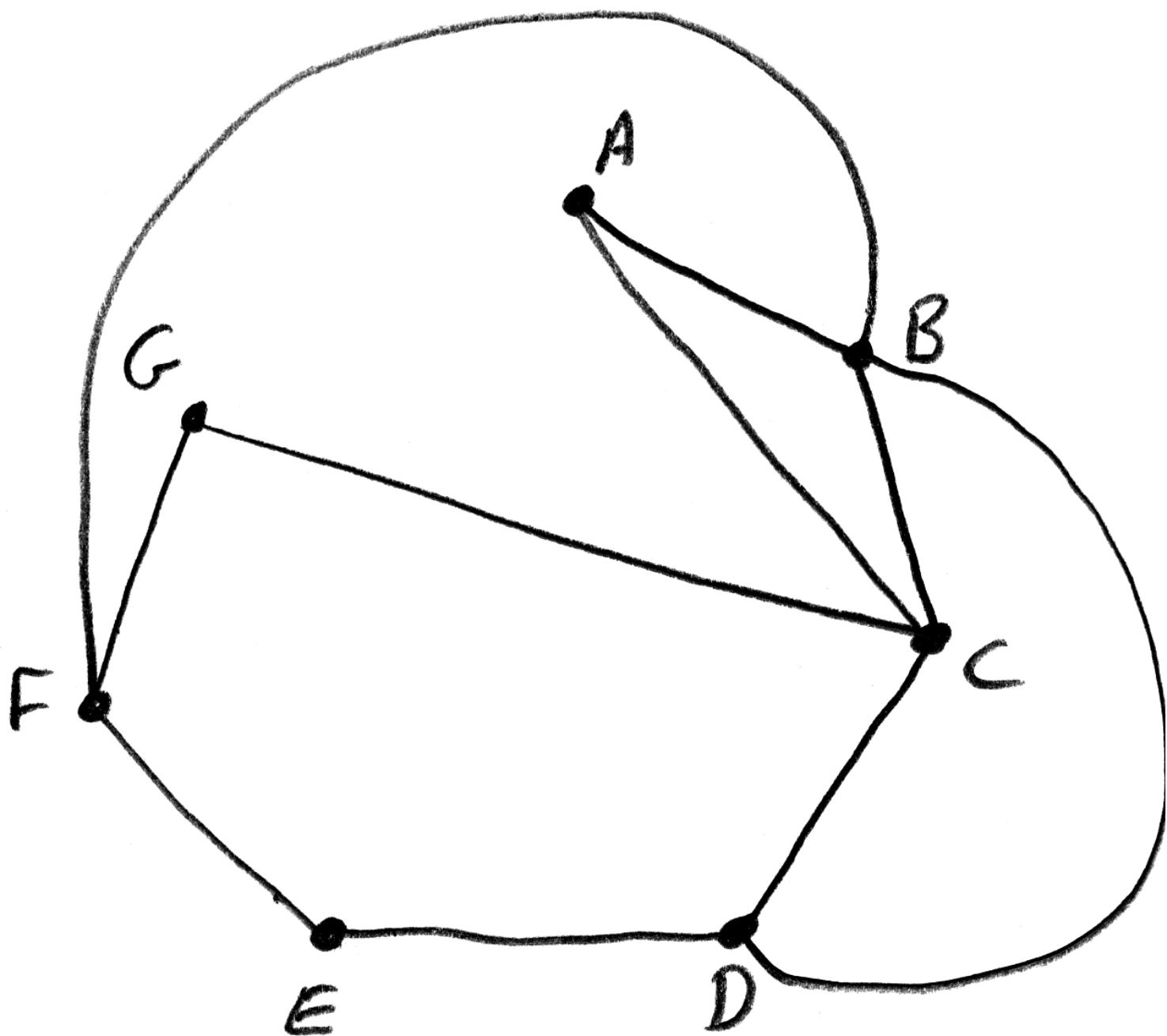
   since S is the sum of even numbers, then it is even   *R1*

   hence T must be even   *R1*

hence there must be an even number of vertices of odd degree   *AG*

[5 marks]

b. (i)
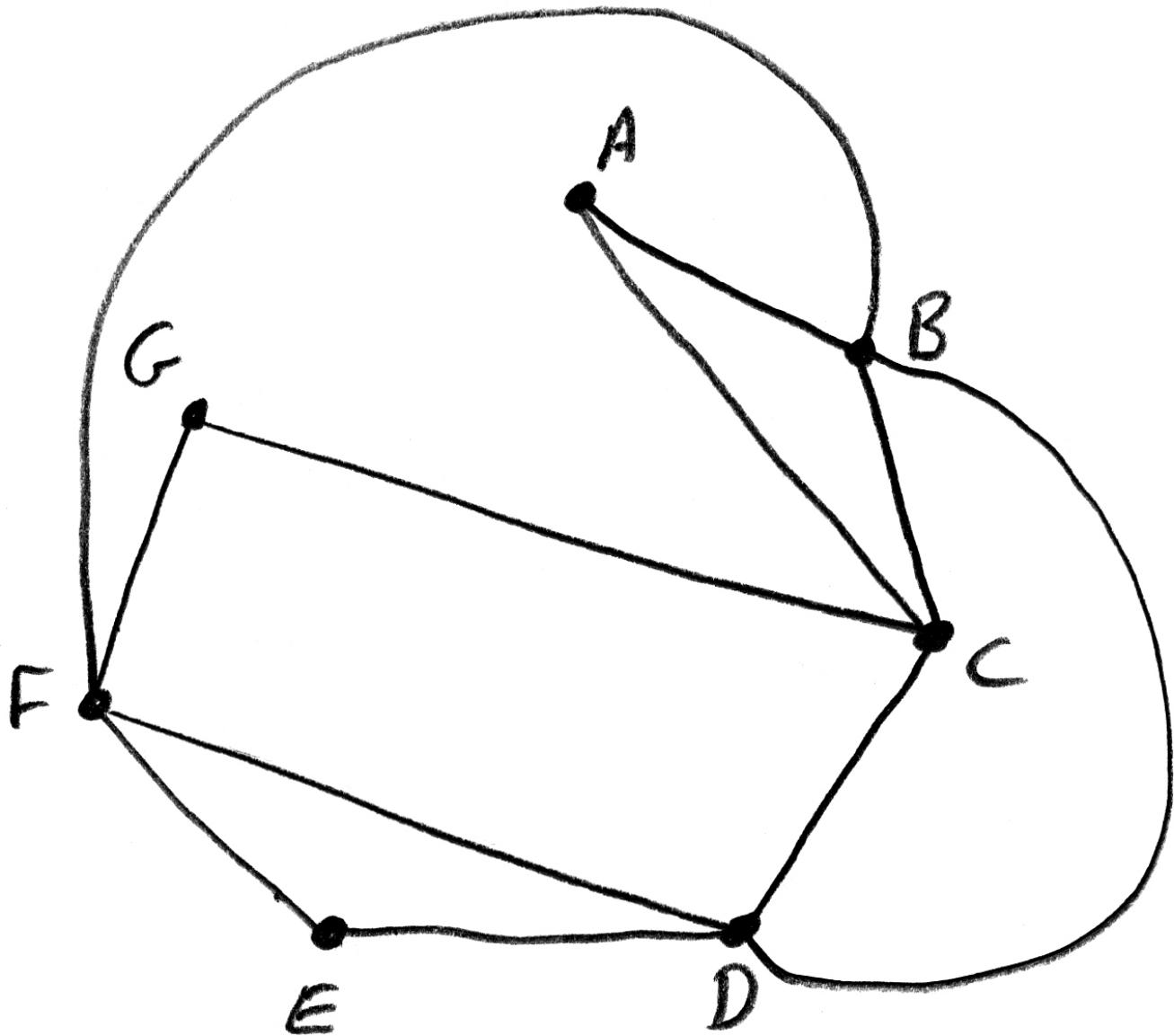
   A1

(ii)   the graph M is not Eulerian because vertices D and F are of odd degree   *A1*

(iii)   the edge which must be added is DF   *A1*

(iv)

a possible Eulerian circuit is ABDFBCDEFGCA   *A2*

Note: award *A1* for a correct Eulerian circuit not starting and finishing at A.

(v)   a Hamiltonian cycle is one that contains each vertex in N *A1*

with the exception of the starting and ending vertices, each vertex must only appear once   *A1*

a possible Hamiltonian cycle is ACGFEDBA   *A1*

(vi)
$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$ *A2*

(vii)   using adjacency matrix to power 4   *(M1)*

C and F   *A1*

[12 marks]

Examiners report

a. Most candidates were able to start (a), but many found problems in expressing their ideas clearly in words. Stronger candidates had little

   problem with (b), but a significant number of weaker candidates had problems working with the concepts of Eulerian circuits and Hamiltonian

   cycles and with understanding how to find a specific number of walks of a certain length as required in (b) (vii).

b. Most candidates were able to start (a), but many found problems in expressing their ideas clearly in words. Stronger candidates had little

   problem with (b), but a significant number of weaker candidates had problems working with the concepts of Eulerian circuits and Hamiltonian

   cycles and with understanding how to find a specific number of walks of a certain length as required in (b) (vii).